Vector Space Approach to Narrowband Interference Excision

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To realize the full performance advantage of a pseudonoise spread-spectrum communication system, a receiver must suppress strong narrowband interferers. Conventional frequency-domain excision techniques are complicated by spreading effects due to the sidelobes of the $\sin(x)/x$ function. This paper shows that such problems can be avoided by regarding the N samples of a band-limited time series and its finite Fourier transform as a vector in an N-dimensional space. Narrowband interference can be suppressed by projecting the data vector onto the subspace orthogonal to the one that best fits the interference. The theory developed here includes a least squares criterion for finding this subspace.

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EXECUTIVE SUMMARY

BACKGROUND

Pseudonoise (PN) spread-spectrum communication systems that distribute signal energy over a wide bandwidth possess low-probability-of-intercept and antijam capabilities. The required power gain for signal detection is obtained using a filter matched to a PN sequence known to the sender and receiver, but not to a potential unfriendly interceptor. Within the spectrum of the wideband signal employed there may lie strong narrowband interferers (including jammers); such interference is particularly troublesome at HF frequencies, where many narrow communication bands with considerable power are present. In order to realize the full performance advantage of such a spread-spectrum system, a receiver must suppress this interference prior to matched filtering with the appropriate PN sequence.

PURPOSE

The estimation and suppression of strong narrowband interference can be carried out through the use of frequency-domain filters in real time. The required finite Fourier transform of the time data is generally performed using basis frequencies that are integer multiples of the reciprocal of the time period over which the measurements are made. As a result of the spreading effects of the sidelobes of the $\sin(x)/x$ function, which is the Fourier transform of a rectangular time window, considerable energy from non-basis frequencies in a narrow band may be distributed over basis frequencies outside the band. Conventional approaches often suppress the sidelobes by weighting the time data before performing the Fourier transform. This results in data distortion prior to the suppression of any of the interference and in the sacrifice of some frequency resolution.

This paper develops an approach in which the N unweighted time-domain values of a sampled band-limited signal plus interference and the N Fourier amplitudes are treated as the components of a vector in an N-dimensional space. The subspace that best fits strong narrowband interference in a least squares sense is excised. Such an approach accounts for and suppresses the sidelobes of a narrowband interferer.

RESULTS

It is shown that one may associate a unit vector in the N-dimensional space mentioned above with each frequency in the continuous range of frequencies less than the Nyquist frequency and may select any subset of N-orthogonal vectors from this set for a basis (coordinate axes). Although the basis may correspond to the usual choice of frequencies that are integer multiples of the sampling frequency divided by N, an infinite

number of possible frequency "reference frames" exist. Any unit frequency vector has components along the coordinate axes that are determined by the central and side lobes of the sin(x)/x function. The projection of the data vector along any unit frequency vector is the finite Fourier transform for that frequency. The "Fourier vector" for any frequency is defined to be the corresponding unit vector multiplied by the Fourier amplitude.

Assume that there are J positive frequency interference bands within which the basis frequency Fourier amplitudes exceed a predetermined noise threshold and a total of K such basis frequencies within these bands, where K ≪ N. (The initial noise threshold should be set high enough to satisfy this requirement -- then lowered after the interference thus identified has been suppressed.) This report approximates the continuous range of frequencies in positive frequency band j by L(j) frequencies, resulting in a total of L frequencies in the J bands, where N ≫ L > 4K. (The lower limit here is established in an appendix.) The subspace $\tilde{V}$ of the N-dimensional space that best fits the Fourier vectors for these L frequencies and the corresponding negative frequencies in a least squares sense is then obtained.

The desired subspace V is found to be determined by the eigenvectors belonging to the M largest eigenvalues of two L × L real symmetric matrices, where M ~ 2K. By projecting the data vector onto the subspace orthogonal to V, the amplitudes in the interference bands are made negligible compared to the noise threshold and the sidelobes of these bands are simultaneously reduced.

CONCLUSIONS

The methods described here should lead to considerable improvement over conventional approaches (i.e., the same amount of narrowband noise excision will lead to much less signal degradation and the same amount of signal degradation will result in much more narrowband noise excision), but the degree of improvement is presently unknown. This enhancement should be particularly useful in a high-noise environment with many closely spaced strong narrowband interferers. It is believed that the processors becoming available will make the required digital processing feasible in real time. Although the vector space approach was developed with a PN spread-spectrum communication system in mind, it should be possible to adapt this approach to other systems.

RECOMMENDATIONS

It is recommended that testing be performed to compare the performance of vector space methods with conventional methods for excising narrowband interference. The specific system and hardware requirements to implement vector space signal processing methods must be identified and compared to the requirements for known frequency-domain and time-domain approaches.

ACKNOWLEDGMENT

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SECTION 1

INTRODUCTION

1.1 BACKGROUND

There is considerable interest in the military applications of pseudo-
noise (PN) spread-spectrum communication systems because of their low-
probability-of-intercept and antijam capabilities [1,2]. At MITRE, in par-
ticular, wideband (~1 MHz) systems of this type are being investigated for
HF communications [3-5]. In such systems the required power gain for
signal detection is obtained using a filter matched to a PN sequence known
to the sender and receiver -- but not to a potential unfriendly intercep-
tor. Although this type of processing reduces the effect of narrowband
interferers, the system performance can be improved significantly by sup-
pressing the most powerful interferers of this kind prior to matched
filtering [6].

Narrowband interference can result from narrowband communication
systems that lie within the band of the spread-spectrum system or may be
caused by intentional jamming. Its estimation and suppression can be
carried out adaptively in the time domain [6,7] or through the use of
frequency-domain filters in real time [8]. The latter approach appears to
be more promising, but the design of suitable interference suppression fil-
ters is subject to problems resulting from the distribution of some of the
energy from a narrow band in sidelobes [9,10]. This paper develops a vec-
tor space approach in which the subspace that best fits strong narrowband
interference in a least squares sense is excised; such an approach suppres-
ses the sidelobes that complicate conventional frequency-domain methods.

An adaptive time-domain technique requires finding the parameters of a
model that is based on assumptions concerning the nature of the interference

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and how slowly it is changing. On the other hand, since the spectrum of the PN sequence is relatively flat across the wide band that is employed, it is easy to recognize strong narrowband interference in the frequency domain -- a filter is designed to suppress the amplitude at all frequencies at which its magnitude exceeds a predetermined noise threshold. At MITRE, analyses and experiments involving HF interference in wide bandwidths have shown that frequency-domain methods can achieve impressive signal-to-interference improvements over time-domain methods [5].

The required finite Fourier transform on the time data is generally performed using basis frequencies that are integer multiples of the reciprocal of the time period over which the measurements are made. Considerable energy from non-basis frequencies in a narrow band is distributed over basis frequencies outside the band as a result of the spreading effect of the sidelobes of the sin(x)/x function, which is the Fourier transform of a rectangular time window. This spreading effect complicates the problem of interpreting the frequency continuum and of designing filters to excise the interference. It is common to suppress the frequency sidelobes by employing time-domain windows that weight the data prior to performing the finite Fourier transform [9,10]. This is accomplished, however, at some sacrifice in frequency resolution and by distorting the data prior to suppressing the interference.

This paper takes a new approach to the interference problem by emphasizing that the continuous range of frequencies

$$-\frac{F_0}{2} \leq f \leq \frac{F_0}{2}$$

in a band-limited time series can be treated equivalently. The fact that there is only a finite number N of time observations spaced at intervals $T_0 = F_0^{-1}$ means that one is working in an N-dimensional vector space. Each

frequency is represented by a specific direction in this space, but only N such frequency vectors are linearly independent. Any subset of N orthogonal frequency vectors can be chosen as coordinate axes; and, although the usual choice is frequencies that are integer multiples of $(NT_0)^{-1}$, there are an infinite number of equivalent frequency "reference frames." Sidelobes result from the projections of vectors associated with other frequencies onto the coordinate axes. The rules for designing filters for narrowband interference suppression should take into account the vector subspace that best fits the continuum of frequencies in such a band and should be independent of the particular frequency coordinate axes chosen. This paper employs a least squares method to determine that subspace and then projects the data vector onto the subspace orthogonal to it. It is worth pointing out that the independence of phenomena from the particular coordinate frame used has played a major role in the development of physics in this century [11].

The discussion in this paper is restricted to the theoretical background and the mathematical formalism for the vector space approach, and examples of its practical application are not given. Although considerable improvement in performance is expected over conventional frequency-domain methods, the signal processing requirements will be greater. The degree of improvement and the specific system and hardware requirements must be identified in future work.

1.2 OUTLINE OF THE PAPER

Sections 2 through 6 are concerned with the development of a vector space theory for a time series and its Fourier transform. Sections 7 through 10 address the problem of considering the subspace spanned by strong narrowband interferers when designing a filter to suppress them.

Finally, Section 11 presents the results and conclusions and makes recommendations for future work. Appendix A evaluates an integral that is approximated by a sum in Section 7 and gives arguments concerning the number of terms that must be included in the sum.

Section 2 considers a band-limited time series (e.g., a radio signal plus the background noise and interference) that is defined over all time and reviews the sampling theorem for such a time series and its Fourier transform. Section 3 shows that the sampled time values can be regarded as the components of a vector in an infinite-dimensional space, with each time corresponding to a different orthogonal basis vector in this space; i.e., a different coordinate axis. A Fourier transform is a transformation to a new basis corresponding to the continuous range of frequencies with magnitudes less than the Nyquist frequency, but does not change the original data vector. Each frequency has a unique vector associated with it that is orthogonal to all other frequency vectors, and the component of the data vector along this direction is the Fourier transform for that frequency. The time and frequency bases are therefore two different ways to express the same vector, i.e., the same information.

Actual measurements are confined to a finite time interval, so Section 4 points out that this is equivalent to mapping a data vector in the infinite-dimensional background space into an N-dimensional space. In fact, N of the time basis vectors in the original space and the corresponding coordinates are mapped directly into the new space, while the remaining basis vectors are mapped into the null (zero) vector and all information about them is lost.

On the other hand, as Section 5 demonstrates, the effect that the mapping onto the N-dimensional space has on the frequency basis vectors is different, since all frequency vectors are changed by this mapping. Every such vector representing a specific frequency in the infinite-dimensional

space has an image in the N-dimensional one and, as mentioned earlier, in the latter space one can choose any subset of N orthogonal frequency vectors as a basis. Section 6 demonstrates how an arbitrary data vector in the N-dimensional space can be expressed as a linear combination of the N vectors in any one of the infinite number of possible frequency bases. The coordinates for the basis chosen can be found by means of a finite Fourier transform. With any frequency one can associate a "Fourier vector," which is the finite Fourier amplitude for that frequency multiplied by the unit vector associated with it, and the N-dimensional data vector can also be expressed as an integral over all the Fourier vectors.

Although in practice only real time series are of interest, Section 7 addresses the problem of excising a single strong narrowband interferer in a complex time series. This is because a "single band" in a real time series actually consists of one in the positive frequency region and another in the negative frequency one, so the problem is first simplified as much as possible. Suppose there are K basis frequencies in the band, such that the Fourier amplitudes for all of these frequencies have magnitudes exceeding the noise threshold N_o. After calculating the Fourier vectors for L frequencies in this band, where L is at least four times larger than K (as discussed in Appendix A), but much less than N, we find the subspace of dimension $M \sim K$ that best fits these L Fourier vectors in a least squares sense. After this subspace is excised by projecting the data vector onto the subspace orthogonal to it, the Fourier amplitudes for frequencies in the band all have magnitudes much less than N_o, and the sidelobes of this band are also suppressed. The required subspace is determined by the eigenvectors corresponding to the M largest eigenvalues of an N × N Hermitian matrix. However, in Section 8, it is demonstrated that this is transformed easily into an L × L real symmetric matrix problem.

Section 9 then generalizes the results to a single interferer in a real time series, and in Section 10 it is shown how to suppress an

arbitrary number of interference bands. The subspace excised is the one that best fits L Fourier vectors in the positive frequency interference bands and the L complex-conjugates of these Fourier vectors. (The latter are included to account for the negative frequencies.) Some expressions are given for the 2L × 2L real symmetric matrix whose largest eigenvalues and corresponding eigenvectors must be found. Finally, it is shown that the problem can be re-expressed in terms of two L × L submatrices.

Note that our treatment of the infinite-dimensional time series from which the finite-dimensional data measurements are taken assumes that the Fourier transform exists, whereas in the literature it is often assumed that such a Fourier transform does not exist [10]. Although such assumptions may require a different theoretical treatment of the mapping from the infinite-dimensional space to the N-dimensional one, the final results should not differ from those in this paper.

SECTION 2

SAMPLING OVER INFINITE TIME

2.1 BAND-LIMITED TIME SERIES

Let $g(t)$ be a real time series (e.g., a radio signal plus background noise) that is a function of the time t and for which the Fourier transform $G(f)$ is defined as a function of the frequency f. Only band-limited functions such that

$$G(f) = 0 \text{ for } |f| > F_0/2 , \tag{2-1}$$

where F_0 is a positive frequency, will be considered in this paper. The Fourier transform and its inverse are given by

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \tag{2-2a}$$

and

$$g(t) = \int_{-F_0/2}^{F_0/2} G(f)e^{j2\pi ft}df , \tag{2-2b}$$

respectively. The fact that $g(t)$ is real leads to the usual condition:

$$G(-f) = G*(f) , \tag{2-3}$$

where the asterisk indicates complex conjugation.

Let $h(t)$ be another such band-limited function with the Fourier transform $H(f)$. The inner product of $g(t)$ and $h(t)$ is defined by

$$(h,g) = (g,h) = \int_{-\infty}^{\infty} h(t)g(t)dt \qquad (2\text{-}4a)$$

$$= \int_{-F_0/2}^{F_0/2} H*(f)G(f)df \ , \qquad (2\text{-}4b)$$

and the norm of g(t) is

$$||g|| = (g,g)^{1/2} \qquad (2\text{-}5a)$$

$$= \left[\int_{-\infty}^{\infty} |g(t)|^2 dt \right]^{1/2} \qquad (2\text{-}5b)$$

$$= \left[\int_{-F_0/2}^{F_0/2} |G(f)|^2 df \right]^{1/2} \ . \qquad (2\text{-}5c)$$

2.2 SINUSOIDAL TIME SERIES

A sinusoid having a frequency F_a such that $F_0/2 > F_a > 0$ can be written in the form

$$\omega_a(t) = A_a \cos(2\pi F_a t - \Phi_a) \qquad (2\text{-}6a)$$

$$= A_a \left[\cos(\Phi_a)\cos(2\pi F_a t) + \sin(\Phi_a)\sin(2\pi F_a t) \right] \ , \qquad (2\text{-}6b)$$

where the phase Φ_a satisfies

$$2\pi > \Phi_a \geq 0 \ ,$$

and where A_a is a real positive constant. The generalization of the Fourier transform in Equation (2-2a) to such a function is

$$\Omega_a(f) = \int_{-\infty}^{\infty} \omega_a(t)e^{-j2\pi ft}dt \tag{2-7a}$$

$$= \frac{A_a}{2}\left[e^{-j\Phi_a}\delta(f - F_a) + e^{j\Phi_a}\delta(f + F_a)\right] , \tag{2-7b}$$

where $\delta(f)$ is the Dirac distribution (which is often incorrectly called the Dirac delta function). In addition, the inner product in Equation (2-4) generalizes to

$$(\omega_a, \omega_b) = \int_{-\infty}^{\infty} \omega_a(t)\omega_b(t)dt \tag{2-8a}$$

$$= (A_a A_b/2)\cos(\Phi_a - \Phi_b)\delta(F_a - F_b) , \tag{2-8b}$$

so two such sinusoids are orthogonal if either

$$F_a \neq F_b$$

or if

$$\Phi_a - \Phi_b = \pm \pi/2 \text{ or } \pm 3\pi/2 .$$

However, since the right-hand side of Equation (2-8b) is not defined for $F_a = F_b$, such functions do not possess norms like that in Equation (5).

2-3

2.3 THE SAMPLING THEOREM

The famous sampling theorem [12] says that it is sufficient to sample a band-limited function such as $G(f)$ at the rate F_0 in order to obtain complete information about it. The sampling interval is

$$T_0 = 1/F_0 \ . \tag{2-9}$$

For the sampling times

$$t = kT_0 \ ; \ k = 0, \pm 1, \pm 2, \ \cdot \ \cdot \ \cdot \ , \tag{2-10}$$

one defines

$$g_k = g(kT_0) \ ; \ k = 0, \pm 1, \pm 2, \ \cdot \ \cdot \ \cdot \ \ . \tag{2-11}$$

Let us also introduce the function

$$\Theta(f) = \begin{cases} 0 \ ; \ f < -F_0/2 \ , \\ 1 \ ; \ -F_0/2 \le f \le F_0/2 \ , \\ 0 \ ; \ F_0/2 < f \ . \end{cases} \tag{2-12}$$

According to the sampling theorem, one can replace Equation (2-2) by

$$G(f) = \Theta(f)T_0 \sum_{k=-\infty}^{\infty} g_k e^{-j2\pi kT_0 f} \ , \tag{2-13a}$$

with the inverse relation

$$g_k = \int_{-F_0/2}^{F_0/2} G(f)e^{j2\pi fkT_0}df \ ; \ k = 0, \pm 1, \pm 2, \ \cdot \ \cdot \ \cdot \ \ . \tag{2-13b}$$

2-4

The inner product in Equation (2-4) can be rewritten in the form

$$(h,g) = T_0 (h,g)_{T_0} , \tag{2-14}$$

where

$$(h,g)_{T_0} = \sum_{k=-\infty}^{\infty} h_k^* g_k \tag{2-15a}$$

$$= \frac{1}{T_0} \int_{-F_0/2}^{F_0/2} H^*(f)G(f)df . \tag{2-15b}$$

The norm in Equation (2-5) becomes

$$||g|| = \sqrt{T_0} \; ||g||_{T_0} , \tag{2-16}$$

where

$$||g||_{T_0} = \left[\sum_{k=-\infty}^{\infty} |g_k|^2 \right]^{1/2} \tag{2-17a}$$

$$= \left[(1/T_0) \int_{-F_0/2}^{F_0/2} |G(f)|^2 df \right]^{1/2} . \tag{2-17b}$$

The sampled time series represented by the sinusoid in Equation (2-6) is

$$\omega_{ak} = \omega_a(kT_0)$$

$$= A_a \cos(2\pi F_a kT_0 - \Phi_a) \ ; \ k = 0, \pm 1, \pm 2, \ldots \tag{2-18}$$

Corresponding to the result in Equation (2-13a), the Fourier transform in Equation (2-7a) can be replaced by the discrete-time transform

$$\Omega_a(f) = \Theta(f)T_0 \sum_{k=-\infty}^{\infty} \omega_{ak} e^{-j2\pi kT_0 f} \ , \tag{2-19}$$

and this is still equal to the right-hand side of Equation (2-7b).

In place of Equation (2-8) one now has

$$(\omega_a, \omega_b) = T_0(\omega_a, \omega_b)_{T_0} \ , \tag{2-20}$$

where

$$(\omega_a, \omega_b)_{T_0} = \sum_{k=-\infty}^{\infty} \omega_{ak} \omega_{bk} \tag{2-21a}$$

$$= \frac{A_a A_b}{2T_0} \cos(\Phi_a - \Phi_b) \delta(F_a - F_b) \ . \tag{2-21b}$$

2.4 COMPLEX TIME SERIES

The real time series g_k in Equation (2-11) can be regarded as a special case of a complex time series

$$x_k = x(kT_0) \; ; \; k = 0, \pm 1, \pm 2, \ldots \; , \tag{2-22}$$

where $x(t)$ is a complex band-limited function of the time. The discrete-time Fourier transform is written

$$X(f) = \Theta(f)T_0 \sum_{k=-\infty}^{\infty} x_k e^{-j2\pi kT_0 f} \; , \tag{2-23a}$$

and the inverse relation is

$$x_k = \int_{-F_0/2}^{F_0/2} X(f)e^{j2\pi fkT_0} \, df \; ; \; k = 0, \pm 1, \pm 2, \ldots \; . \tag{2-23b}$$

Since, in general,

$$x_k^* \neq x_k \; ; \; k = 0, \pm 1, \pm 2, \ldots \; , \tag{2-24a}$$

in contrast to Equation (2-3), one usually has

$$X(-f) \neq X*(f) \; . \tag{2-24b}$$

The generalization of the inner product in Equation (2-15) for any two band-limited complex time series x_k and y_k is

$$(y,x)_{T_0} = \left[(x,y)_{T_0} \right]^* = \sum_{k=-\infty}^{\infty} y_k^* x_k \tag{2-25a}$$

$$= \frac{1}{T_0} \int_{-F_0/2}^{F_0/2} Y*(f)X(f)df \; . \tag{2-25b}$$

2-7

SECTION 3

INFINITE-DIMENSIONAL VECTOR SPACE FORMALISM

3.1 TIME REPRESENTATION IN C^∞

The preceding section discussed the values g_k of a real band-limited time function $g(t)$ sampled at times kT_0, $k = 0$, ± 1, ± 2, . . . , where $1/2T_0$ is the Nyquist frequency. It is possible to regard the g_k's as the components of an infinite-dimensional real vector g [13,14], i.e.,

$$g = (\ \cdot \ \cdot \ \cdot \ , \ g_{-1}, \ g_0, \ g_1, \ \cdot \ \cdot \ \cdot \ ) \ . \tag{3-1}$$

The inner product in Equation (2-15) can be viewed as the scalar (or dot) product of two such vectors [13,14]:

$$(h,g)_{T_0} = h^* \cdot g = \sum_{k=-\infty}^{\infty} h_k^* g_k \ . \tag{3-2}$$

We shall refer to this infinite-dimensional vector space with the scalar product defined in Equation (3-2) as R^∞.

If only the time representation of such a series were going to be considered, there would be no need to introduce a complex vector space, since one measures only real time series in practice. But, in order to also be able to consider the usual frequency representation, in which there occur complex amplitudes, one must work in an infinite-dimensional complex vector space C^∞. The complex time series in Equation (2-22) defines an infinite-dimensional complex vector

$$x = (. \ \cdot \ \cdot \ \cdot \ , \ x_{-1}, \ x_0, \ x_1, \ \cdot \ \cdot \ \cdot \ ) \ . \tag{3-3}$$

From Equation (2-25a), the scalar (or dot product) is

$$(y,x)_{T_0} = \left[(x,y)_{T_0}\right]^* = y^* \cdot x \tag{3-4a}$$

$$= \sum_{k=-\infty}^{\infty} y_k^* x_k \; . \tag{3-4b}$$

The norm of x can thus be written:

$$||x||_{T_0} = \left[x^* \cdot x\right]^{1/2} = \left[\sum_{k=-\infty}^{\infty} |x_k|^2\right]^{1/2} \; . \tag{3-5}$$

Let us introduce a set of real orthonormal vectors

$$e(k) = (\; . \; . \; . \; , \; e_{-1}(k) \; , \; e_0(k) \; , \; e_1(k) \; , \; . \; . \; . \;) \; ;$$

$$k = 0, \pm 1, \pm 2, \; . \; . \; . \; , \tag{3-6a}$$

in C^∞ such that

$$e_j(k) = \delta_{jk} \; ; \; j, \; k = 0, \pm 1, \pm 2, \; . \; . \; . \; . \tag{3-6b}$$

Thus, $e(k)$, for a specific value of k, is the vector with a one in the kth position and a zero in all other positions; such a vector represents an infinite sampled time series with the value 1 at time $t = kT_0$ and the value 0 at all other times $t = lT_0$ with $l \neq k$. The vectors in Equation (3-6) are orthonormal because

$$(e(j), e(k))_{T_0} = e(j) \cdot e(k) = \delta_{jk} \; ;$$

$$j, \; k = 0, \; \pm 1, \; \pm 2, \; \ldots \; . \tag{3-7}$$

For an arbitrary vector $\mathbf{x}$ in C^{∞}, Equation (3-3) can now be written:

$$\mathbf{x} = \sum_{k=-\infty}^{\infty} e(k) x_k \; , \tag{3-8a}$$

with the inverse

$$x_k = e(k) \cdot \mathbf{x} \; ; \; k = 0, \; \pm 1, \; \pm 2, \; \ldots \; . \tag{3-8b}$$

The infinite set of linearly independent vectors in Equation (3-6) is said to form an orthonormal basis for C^{∞}, and the x_k's are the coordinates in this basis. For a particular value of k one can call $e(k)$ the unit vector in the k-direction. The complex conjugate

$$\mathbf{x}^{\star} = \sum_{k=-\infty}^{\infty} e(k) x_k^{\star} \tag{3-9}$$

of the vector $\mathbf{x}$ is also a vector in this space. Of course, for a real time series represented by the vector $\mathbf{g}$ in Equation (3-1),

$$\mathbf{g}^{\star} = \mathbf{g} = \sum_{k=-\infty}^{\infty} e(k) g_k \; , \tag{3-10a}$$

that is,

$$g_k^* = g_k \; ; \; k = 0, \, \pm 1, \, \pm 2, \, \ldots \, . \tag{3-10b}$$

3.2 FREQUENCY REPRESENTATION IN C^∞

In order to have a basis corresponding to the complex Fourier ampli-
tude $X(f)$ in Equation (2-23a), let us define the complex vectors $\varepsilon(f)$ by

$$\varepsilon(f) = \Theta(f) \sum_{k=-\infty}^{\infty} e(k) e^{j2\pi k T_0 f} \; . \tag{3-11a}$$

The inverse transformation, which is similar in form to Equation (2-23b),
is

$$e(k) = T_0 \int_{-F_0/2}^{F_0/2} \varepsilon(f) e^{-j2\pi k T_0 f} df \; ; \; k = 0, \, \pm 1, \, \pm 2, \, \ldots \, . \tag{3-11b}$$

The vectors $\varepsilon(f)$ satisfy the relations

$$\varepsilon(-f) = \varepsilon^*(f) \tag{3-12a}$$

and

$$(\varepsilon(f'), \varepsilon(f))_{T_0} = \varepsilon^*(f') \cdot \varepsilon(f) = \Theta(f)\Theta(f')F_0 \, \delta(f-f') \; . \tag{3-12b}$$

From a comparison of Equations (3-8a) and (3-11a), it is clear that, for
$F_0/2 \geq |f| \geq 0$, $\varepsilon(f)$ represents the complex time series

$$\varepsilon_k(f) = e^{j2\pi k T_0 f} \; ; \; k = 0, \, \pm 1, \, \pm 2, \, \ldots \, . \tag{3-13}$$

The vectors $\varepsilon(f)$ are also a basis for C^∞, since when Equations (2-23b) and (3-11b) are substituted into Equation (3-8a), the result is

$$x = \int_{-F_0/2}^{F_0/2} \varepsilon(f)X(f)df \, , \qquad (3\text{-}14a)$$

with the inverse

$$X(f) = T_0 \varepsilon^\star(f) \cdot x \, . \qquad (3\text{-}14b)$$

Equations (3-8a) and (3-14a) are two different ways in which to represent the same vector x in C^∞. Just as the sampled time values x_k are the coordinates of x in the basis $e(k)$; $k = 0, \pm 1, \pm 2, \ldots$, the Fourier amplitudes $X(f)$ are the coordinates in the basis $\varepsilon(f)$ for $F_0/2 \geq f \geq -F_0/2$. Note that when x is expressed as in Equation (3-14a), we have, in place of Equation (3-4),

$$(y,x)_{T_0} = y^\star \cdot x = \frac{1}{T_0} \int_{-F_0/2}^{F_0/2} Y^\star(f)X(f)df \, , \qquad (3\text{-}15)$$

as in Equation (2-25b).

3.3 POSITIVE AND NEGATIVE FREQUENCY DECOMPOSITION

An arbitrary complex vector x can be written

$$x = x(+) + x(-) \, , \qquad (3\text{-}16)$$

where

$$x(+) = \int_0^{F_0/2} \varepsilon(f)X(f)df \qquad (3\text{-}17a)$$

contains only positive frequencies and

$$x(-) = \int_{-F_0/2}^{0} \varepsilon(f)X(f)df \qquad (3\text{-}17b)$$

$$= \int_0^{F_0/2} \varepsilon^\star(f)X(-f)df \qquad (3\text{-}17c)$$

contains only negative ones. In general, the vectors $x(+)$ and $x(-)$ are independent of each other, and they are always orthogonal, i.e.,

$$(x(-),x(+))_{T_0} = x(-)^\star \cdot x(+) = 0 \ . \qquad (3\text{-}18)$$

The result in Equation (3-14) is, of course, still valid when the time series under consideration is represented by a real vector g, as in Equation (3-10), but the Fourier amplitude $G(f)$ is now subject to the constraint in Equation (2-3). One can write

$$g = \int_{-F_0/2}^{F_0/2} \varepsilon(f)G(f)df \ , \qquad (3\text{-}19a)$$

with the inverse

$$G(f) = T_0\, \varepsilon^\star(f) \cdot g \ . \qquad (3\text{-}19b)$$

3-6

We can still decompose **g** into a positive frequency part **g**(+) and a negative frequency part **g**(-), as in Equations (3-16) and (3-17):

$$\mathbf{g} = \mathbf{g}(+) + \mathbf{g}(-) \ . \tag{3-20}$$

But now, because of Equation (2-3),

$$\mathbf{g}(-) = \mathbf{g}(+)\star \ . \tag{3-21}$$

3.4   REPRESENTATION OF A SINUSOID

To conclude this section, let us consider a real vector $\boldsymbol{\omega}_a$ in C^∞ representing the sinusoid in Equation (2-18).   Thus,

$$\boldsymbol{\omega}_a = \sum_{k=-\infty}^{\infty} e(k)\omega_{ak} \tag{3-22a}$$

$$= A_a \sum_{k=-\infty}^{\infty} e(k)\cos(2\pi F_a k T_0 - \Phi_a) \tag{3-22b}$$

$$= \frac{A_a}{2} \left[e^{-i\Phi_a} \varepsilon(F_a) + e^{j\Phi_a} \varepsilon\star(F_a) \right] \ . \tag{3-22c}$$

The scalar product of two such sinusoids can then be written

$$(\boldsymbol{\omega}_a, \boldsymbol{\omega}_b)_{T_0} = \boldsymbol{\omega}_a \cdot \boldsymbol{\omega}_b$$

$$= \frac{A_a A_b}{2T_0} \cos(\Phi_a - \Phi_b)\delta(F_a - F_b) \ , \tag{3-23}$$

as in Equation (2-21).

The vector $\boldsymbol{\omega}_a$ can be decomposed into a positive and a negative frequency part as in Equation (3-20), i.e.,

$$\boldsymbol{\omega}_a = \boldsymbol{\omega}_a(+) + \boldsymbol{\omega}_a(-) \, , \qquad\qquad (3\text{-}24)$$

where

$$\boldsymbol{\omega}_a(\pm) = A_a(\pm)\,\mathcal{E}(\pm F_a) \qquad\qquad (3\text{-}25a)$$

$$= \sum_{k=-\infty}^{\infty} e(k)\boldsymbol{\omega}_{ak}(\pm) \, . \qquad\qquad (3\text{-}25b)$$

Here we have used

$$A_a(\pm) = \frac{A_a}{2}\, e^{\mp j\Phi_a} \, , \qquad\qquad (3\text{-}26a)$$

$$\boldsymbol{\omega}_{ak}(\pm) = A_a(\pm) e^{\pm j 2\pi F_a k T_0} \, . \qquad\qquad (3\text{-}26b)$$

These vectors satisfy

$$\boldsymbol{\omega}_a(-)^\star = \boldsymbol{\omega}_a(+) \qquad\qquad (3\text{-}27)$$

and

$$(\boldsymbol{\omega}_a(\pm),\boldsymbol{\omega}_b(\pm))_{T_0} = \boldsymbol{\omega}_a(\pm)^\star \cdot \boldsymbol{\omega}_b(\pm) = A_a(\pm)^\star A_b(\pm)\delta(F_a - F_b) \, , \qquad\qquad (3\text{-}28a)$$

$$(\boldsymbol{\omega}_a(+),\boldsymbol{\omega}_b(-))_{T_0} = \boldsymbol{\omega}_a(+)^\star \cdot \boldsymbol{\omega}_b(-) = 0 \, . \qquad\qquad (3\text{-}28b)$$

SECTION 4

FINITE TIME SERIES

4.1 TIME REPRESENTATION IN C^N

Actual measurements of a band-limited time series are made over a finite time interval and not over infinite time, as the sampling theorem assumes. Since there is only a finite number N of samples, one is measuring the components of a vector in an N-dimensional space. This section discusses the time-domain mapping from the infinite-dimensional vector space C^∞ considered in Section 3 to an N-dimensional one, and Sections 5 and 6 discuss the effect in the frequency domain.

Suppose we are interested in the complex time series

$$x_k \; ; \; k = 0, \pm 1, \pm 2, \ldots ,$$

discussed earlier, but have knowledge of only N successive values, say,

$$x_k \; ; \; 0 \leq k \leq N - 1 , \tag{4-1}$$

where it is assumed that N is even. These values can be regarded as the components of a vector $\hat{x}$ in an N-dimensional complex vector space C^N [13,14]:

$$\hat{x} = (x_0, x_1, \ldots, x_{N-1}) . \tag{4-2}$$

Just as the scalar product in C^∞ was defined in Equation (3-4), the scalar product of two vectors $\hat{x}$ and $\hat{y}$ in C^N is [13,14]

$$(\hat{y}, \hat{x})_N = (\hat{x}, \hat{y})_N^* = \hat{y}^* \cdot \hat{x} \tag{4-3a}$$

$$= \sum_{k=0}^{N-1} y_k^* x_k \; , \tag{4-3b}$$

where the components of $\hat{y}$,

$$y_k \; ; \; 0 \leq k \leq N - 1 \; ,$$

are also taken from an infinite time series.

The norm or length of $\hat{x}$ is

$$||\hat{x}||_N = \left[\hat{x}^* \cdot \hat{x} \right]^{1/2} = \left[\sum_{k=0}^{N-1} |x_k|^2 \right]^{1/2} \; . \tag{4-4}$$

For a basis in C^N one may employ the real vectors

$$\hat{e}(0) = (1,0,0, \; . \; . \; . \; ,0) \; ,$$

$$\hat{e}(1) = (0,1,0, \; . \; . \; . \; ,0) \; ,$$
$$\cdot$$
$$\cdot \tag{4-5}$$
$$\cdot$$
$$\hat{e}(N-1) = (0,0,0, \; . \; . \; . \; ,1) \; .$$

These satisfy the properties required of an orthonormal basis, i.e.,

$$(\hat{e}(j),\hat{e}(k))_N = \hat{e}(j) \cdot \hat{e}(k) = \delta_{jk} \; ; \; 0 \leq j,k \leq N - 1 \; . \tag{4-6}$$

Thus, $\hat{x}$ may be written:

$$\hat{x} = \sum_{k=0}^{N-1} \hat{e}(k)x_k \ , \tag{4-7a}$$

with the inverse

$$x_k = \hat{e}(k) \cdot \hat{x} \ . \tag{4-7b}$$

4.2 MAPPING OF VECTORS FROM C^∞ TO C^N

One can formally define a map M from C^∞ to C^N by requiring that

$$e(k) \xrightarrow{\ M\ } \begin{cases} \hat{e}(k) \ ; \ 0 \leq k \leq N - 1 \ , \\ 0 \quad \ ; \ k \geq N \text{ or } k < 0 \ . \end{cases} \tag{4-8}$$

An operator $\tilde{M}$ that will perform this map is defined by

$$\tilde{M} = \sum_{j=0}^{N-1} \ \sum_{k=-\infty}^{\infty} \hat{e}(j)m_{jk}e(k) \ . \tag{4-9a}$$

where

$$m_{jk} = \begin{cases} \delta_{jk} \ ; \ 0 \leq j,k \leq N - 1 \ , \\ 0 \quad \ ; \ k \geq N \text{ or } k < 0 \ . \end{cases} \tag{4-9b}$$

Another way to write $\tilde{M}$ is simply

$$\tilde{M} = \sum_{k=0}^{N-1} \hat{e}(k)e(k) \ . \tag{4-10}$$

This operator transforms a basis vector in C^{∞} to one in C^N in the following way:

$$\tilde{M} \cdot e(n) = \left[\sum_{j=0}^{N-1} \sum_{k=-\infty}^{\infty} \hat{e}(j) m_{jk} e(k) \right] \cdot e(n)$$

$$= \sum_{j=0}^{N=1} \hat{e}(j) \sum_{k=-\infty}^{\infty} m_{jk} \left[e(k) \cdot e(n) \right]$$

$$= \sum_{j=0}^{N-1} \hat{e}(j) \sum_{k=-\infty}^{\infty} m_{jk} \delta_{kn} \ ,$$

so, when use is made of Equation (4-9b), we find that

$$\tilde{M} \cdot e(n) = \begin{cases} \hat{e}(n) \ ; \ 0 \le n \le N - 1 \ , \\ 0 \quad \ ; \ n \ge N \ \text{or} \ n < 0 \ , \end{cases} \tag{4-11}$$

which is the desired result.

For an arbitrary vector x in C^{∞} having the form in Equation (3-8a),

$$\tilde{M} \cdot x = \tilde{M} \cdot \left[\sum_{k=-\infty}^{\infty} e(k) x_k \right]$$

4-4

$$= \sum_{k=-\infty}^{\infty} \left[\tilde{\mathbf{M}} \cdot e(k) \right] x_k \ .$$

With the aid of Equation (4-11), the result is

$$\tilde{\mathbf{M}} \cdot \mathbf{x} = \hat{\mathbf{x}} \ , \tag{4-12}$$

where $\hat{\mathbf{x}}$ has the form in Equation (4-7a). We can say that $\tilde{\mathbf{M}}$ is an operator from C^{∞} to C^N -- note that it does not affect the N components of $\mathbf{x}$ given in Equation (4-1), but it destroys all information about the remaining components in C^{∞}.

The map from C^{∞} to C^N is not one-to-one. Suppose $\mathbf{x}$ and $\mathbf{x}'$ are two different vectors in C^{∞} having the property

$$x_k - x_k' = \begin{cases} 0 \ ; \ 0 \leq k \leq N - 1 \ , \\ \text{arbitrary} \ ; \ k \geq N \ \text{or} \ k < 0 \ . \end{cases} \tag{4-13a}$$

Then

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}' \ , \tag{4-13b}$$

so there is no way to distinguish between $\mathbf{x}$ and $\mathbf{x}'$ using only the N measured components of the time series -- $\mathbf{x}$ and $\mathbf{x}'$ have the same image in C^N. In fact, given any vector $\mathbf{x}$ in C^{∞}, there is an infinite number of vectors $\mathbf{x}'$ in C^{∞} such that

$$\mathbf{x}' \neq \mathbf{x} \ ,$$

but with the same image in C^N as $\mathbf{x}$.

The above treatment is still valid when only real vectors having the form of $\mathbf{g}$ in Equation (3-10a) are considered. Thus

$$\hat{\mathbf{g}} = \tilde{\mathbf{M}} \cdot \mathbf{g} \qquad (4\text{-}14a)$$

$$= \sum_{k=0}^{N-1} \hat{\mathbf{e}}(k) g_k \qquad (4\text{-}14b)$$

is a real vector in C^N. There is an infinite number of real vectors $\mathbf{g}'$ in C^∞ such that

$$\mathbf{g}' \neq \mathbf{g} , \qquad (4\text{-}15a)$$

but for which

$$\hat{\mathbf{g}}' = \hat{\mathbf{g}} . \qquad (4\text{-}15b)$$

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are any two vectors in C^∞ that are orthogonal; i.e., which satisfy

$$(\mathbf{y},\mathbf{x})_{T_0} = 0 . \qquad (4\text{-}16a)$$

In general,

$$(\hat{\mathbf{y}},\hat{\mathbf{x}})_N \neq 0 , \qquad (4\text{-}16b)$$

i.e., the images of $\mathbf{x}$ and $\mathbf{y}$ in C^N are not necessarily orthogonal. On the other hand, suppose that $\mathbf{x}$ and $\mathbf{y}$ are not orthogonal in C^∞, i.e.,

$$(\mathbf{y},\mathbf{x})_{T_0} \neq 0 . \qquad (4\text{-}17a)$$

It is possible that

$$(\hat{y}, \hat{x})_N = 0 \ .$$

(4-17b)

4.3 OPERATORS IN C^N

An arbitrary operator $\tilde{B}_N$ in C^N has the form

$$\tilde{B}_N = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \hat{e}(j) b_{jk} \hat{e}(k) \ ,$$

(4-18)

where the numbers

$$b_{jk} \ ; \ 0 \le j, \ k \le N - 1 \ ,$$

which are generally complex, can be regarded as the elements of an $N \times N$ matrix:

$$\tilde{b} = \left[b_{jk} \right] \ .$$

(4-19)

When $\tilde{B}_N$ operates on an arbitrary vector $\hat{x}$ in C^N, it is easy to verify that one obtains a new vector $\hat{y}$ in C^N,

$$\hat{y} = \tilde{B}_N \cdot \hat{x} \ ,$$

(4-20)

that has the form

$$\hat{y} = \sum_{k=0}^{N-1} \hat{e}(k) y_k \ ,$$

(4-21a)

where

$$y_k = \sum_{l=0}^{N-1} b_{kl} x_l \;\; ; \;\; 0 \leq k \leq N - 1 \; . \tag{4-21b}$$

If the elements of the matrix $\tilde{b}$ are real, then $\tilde{B}_N$ is a real operator and, given an arbitrary real vector $\hat{g}$ in C^N, the vector

$$\hat{h} = \tilde{B}_N \cdot \hat{g} \tag{4-22}$$

is also real.

A particular example of such an operator is

$$\tilde{I}_N = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \hat{e}(j) \delta_{jk} \hat{e}(k) \tag{4-23a}$$

$$= \sum_{k=0}^{N-1} \hat{e}(k) \hat{e}(k) \; . \tag{4-23b}$$

It is easy to verify that, given any vector $\hat{x}$ in C^N,

$$\tilde{I}_N \cdot \hat{x} = \hat{x} \; , \tag{4-24}$$

so $\tilde{I}_N$ is the identity operator in C^N.

SECTION 5

FREQUENCY BASES FOR A FINITE TIME SERIES

5.1 FREQUENCY BASIS MAPPING FROM C^{∞} to C^{N}

The results of the preceding section will be rewritten in the fre-
quency representation, and the consequences that a mapping onto a finite
time interval has in the frequency domain will be analyzed. In this sec-
tion the discussion will be confined to the basis vectors in the frequency
domain, and in the next section the frequency representation of an arbi-
trary vector will be considered.

Equation (3-11a) defines the frequency-domain basis vectors $\varepsilon(f)$ in
C^{∞}. If such a vector is mapped to C^{N}, using the operator $\tilde{M}$ defined in the
preceding section, we obtain

$$\tilde{\varepsilon}(f) = \tilde{M} \cdot \varepsilon(f) \tag{5-1a}$$

$$= \Theta(f) \sum_{k=0}^{N-1} \hat{e}(k) e^{j2\pi k T_0 f} \ . \tag{5-1b}$$

It follows that the scalar product of two such vectors is

$$(\tilde{\varepsilon}(f'), \tilde{\varepsilon}(f))_N = \tilde{\varepsilon}^*(f') \cdot \tilde{\varepsilon}(f) \tag{5-2a}$$

$$= N\Theta(f)\Theta(f')D(f' - f) \ , \tag{5-2b}$$

where

$$D(f) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j2\pi k T_0 f} \qquad (5\text{-}3)$$

is a cyclic function of f with period F_0.

The function D(f) can be written in the form

$$D(f) = \exp[-j(N-1)\pi T_0 f]D^\circ(f) \ , \qquad (5\text{-}4a)$$

where

$$D^\circ(f) = \frac{\sin(\pi N T_0 f)}{N \sin(\pi T_0 f)} \qquad (5\text{-}4b)$$

approximates the behavior of the function $\sin(x)/x$ in an N dimensional space. This function has the properties

$$D(-f) = D*(f) \qquad (5\text{-}5)$$

and

$$NT_0 \int_{-F_0/2}^{F_0/2} D(f - f'')D(f'' - f')df'' = D(f - f') \ . \qquad (5\text{-}6)$$

It has a maximum at f = 0, where

$$D^\circ(0) = D(0) = 1 \ , \qquad (5\text{-}7a)$$

and, more generally,

$$D^o(kF_0/N) = \sum_{l=-\infty}^{\infty} \delta_{k,lN} \; ; \; k = 0, \pm 1, \pm 2, \ldots \; . \tag{5-7b}$$

It is worth noting that

$$D^o(f) = \sum_{n=-\infty}^{\infty} U(f + nF_0) \; , \tag{5-8a}$$

where

$$U(f) = \frac{\sin \pi N T_0 f}{\pi N T_0 f} \tag{5-8b}$$

is the Fourier transform of

$$u(t) = \begin{cases} 0 & ; \; t < -\dfrac{NT_0}{2} \; , \\[2ex] \dfrac{1}{NT_0} & ; \; -\dfrac{NT_0}{2} \leq t \leq \dfrac{NT_0}{2} \; , \\[2ex] 0 & ; \; \dfrac{NT_0}{2} < t \; . \end{cases} \tag{5-8c}$$

That is, $D^o(f)$ is a frequency-aliased [12] version of $U(f)$, which results because the function $u(t)$, which is not band-limited, is sampled at the rate F_0. Since other properties of $D^o(f)$ have been frequently discussed in the literature [9], we will not go into much detail here or provide any figures. The region

$$-\frac{F_0}{N} < f < \frac{F_0}{N}$$

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is frequently called the main lobe of $D^\circ(f)$, while the kth sidelobes are given by

$$k\frac{F_0}{N} < |f| < (k + 1)\frac{F_0}{N} \, ,$$

where

$$1 \leq k \leq \frac{N}{2} - 1 \, .$$

5.2 UNIT FREQUENCY VECTORS IN C^N

Whereas the vectors $\varepsilon(f)$ and $\varepsilon(f')$ in C^N are always orthogonal for $f \neq f'$, their images $\mathbf{\mathcal{E}}(f)$ and $\mathbf{\mathcal{E}}(f')$ in C^N do not generally have this property, according to Equation (5-2). In fact, from Equation (5-7b) it follows that $\mathbf{\mathcal{E}}(f)$ and $\mathbf{\mathcal{E}}(f')$ are orthogonal only when

$$f - f' = \frac{kF_0}{N} \, ; \; k = \pm\, 1, \, \pm\, 2, \, . \, . \, . \, , \, \pm\, (N - 1) \, . \tag{5-9}$$

From Equations (5-2) and (5-7a), the length of $\mathbf{\mathcal{E}}(f)$ is

$$||\mathbf{\mathcal{E}}(f)||_N = \left[\mathbf{\mathcal{E}}^\star(f)\cdot\mathbf{\mathcal{E}}(f)\right]^{1/2} = \sqrt{N} \, . \tag{5-10}$$

Since the scalar product of $\mathbf{\mathcal{E}}(f)$ and $\mathbf{\mathcal{E}}(-f)$ is

$$(\mathbf{\mathcal{E}}(f),\mathbf{\mathcal{E}}(-f))_N = \mathbf{\mathcal{E}}^\star(f)\cdot\mathbf{\mathcal{E}}(-f) = N\Theta(f)D(2f) \, , \tag{5-11}$$

$\mathbf{\mathcal{E}}(f)$ and $\mathbf{\mathcal{E}}(-f)$ are not, in general, orthogonal.

It was pointed out in the previous section that any vector in C^N is the image of an infinite number of vectors in C^∞. Since there is an infinite number of different vectors in C^∞ that, like $\varepsilon(f)$, map into $\hat{\varepsilon}(f)$, frequency loses the unique and unambiguous meaning that it has in infinite time. For example, one of the vectors $\varepsilon'(f) \neq \varepsilon(f)$ that maps into $\hat{\varepsilon}(f)$ is

$$\varepsilon'(f) = \Theta(f) \sum_{k=0}^{N-1} e(k) e^{j2\pi k T_0 f} \tag{5-12a}$$

$$= \Theta(f) N T_0 \int_{-F_0/2}^{F_0/2} \varepsilon(f') D(f' - f) df' \ . \tag{5-12b}$$

Instead of using the vectors $\hat{\varepsilon}(f)$, it is convenient to introduce unit vectors $\hat{v}(f)$ defined by

$$\hat{v}(f) = \frac{1}{\sqrt{N}} \hat{\varepsilon}(f) \tag{5-13a}$$

$$= \frac{1}{\sqrt{N}} \bar{M} \cdot \varepsilon(f) \tag{5-13b}$$

$$= \Theta(f) \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{e}(k) e^{j2\pi k T_0 f} \ . \tag{5-13c}$$

These are unit vectors, since

$$||\hat{v}(f)||_N = 1 \ . \tag{5-14}$$

Furthermore, the scalar product of two such vectors is

$$(\hat{v}(f'), \hat{v}(f))_N = \Theta(f)\Theta(f')D(f' - f) , \qquad (5\text{-}15)$$

so $D(f - f')$ can be regarded as the complex cosine of the angle between $\hat{v}(f')$ and $\hat{v}(f)$. These vectors satisfy

$$\hat{v}(-f) = \hat{v}^*(f) \qquad (5\text{-}16a)$$

and

$$(\hat{v}(f), \hat{v}(-f))_N = \Theta(f)D(2f) . \qquad (5\text{-}16b)$$

5.3 ORTHONORMAL FREQUENCY BASES IN C^N

Since C^N is an N-dimensional space, we can choose N orthogonal vectors from the set

$$\hat{v}(f) ; \quad |f| \leq \frac{F_0}{2}$$

as an orthonormal basis -- there is an infinite number of possibilities. To obtain all possible orthonormal bases, let us first consider the set of frequencies

$$f_k = \begin{cases} k\,\dfrac{F_0}{N} ; \ 0 \leq k \leq \dfrac{N}{2} - 1 , \\[2em] -(N - k)\dfrac{F_0}{N} ; \ \dfrac{N}{2} \leq k \leq N - 1 . \end{cases} \qquad (5\text{-}17a)$$

Note that

$$f_{N-k} = -f_k \ ; \ 1 \leq k \leq \frac{N}{2} - 1 \ . \tag{5-17b}$$

Now consider the more general set of frequencies

$$\overline{f}_k = \overline{f} + f_k \ ; \ 0 \leq k \leq N - 1 \ , \tag{5-18a}$$

where $\overline{f}$ has a value in the range

$$\frac{F_0}{N} > \overline{f} \geq 0 \ . \tag{5-18b}$$

From Equations (5-7) and (5-15) the set

$$\hat{v}(\overline{f}_k) \ ; \ 0 \leq k \leq N - 1 \ , \tag{5-19a}$$

is an orthonormal basis; i.e.,

$$(\hat{v}(\overline{f}_j), \hat{v}(\overline{f}_k))_N = \hat{v}^*(\overline{f}_j) \cdot \hat{v}(\overline{f}_k) = \delta_{jk} \ ; \ 0 \leq j,k \leq N - 1 \ . \tag{5-19b}$$

Because of Equation (5-13c),

$$\hat{v}(\overline{f}_k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{e}(n) e^{j2\pi n T_0 \overline{f}_k} \ ; \ 0 \leq k \leq N - 1 \ . \tag{5-20a}$$

The inverse of this equation is

$$\hat{e}(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{v}(\overline{f}_k) e^{-j2\pi n T_0 \overline{f}_k} \; ; \; 0 \leq n \leq N - 1 \; . \tag{5-20b}$$

With the aid of Equation (5-15), it is easy to show that, for an arbitrary choice of basis frequencies,

$$\hat{v}(f) = \Theta(f) \sum_{k=0}^{N-1} \hat{v}(\overline{f}_k) D(\overline{f}_k - f) \; . \tag{5-21}$$

This relation enables us to express the unit vector $\hat{v}(f)$, which is parallel to the mapping of the frequency vector $\varepsilon(f)$ in C^∞ into the space C^N, in terms of the N unit vectors $\hat{v}(\overline{f}_k)$ chosen for a basis. Here $D(\overline{f}_k - f)$ is the complex cosine of the angle between the basis vector $\hat{v}(\overline{f}_k)$ and the arbitrary frequency vector $\hat{v}(f)$. A consequence of Equations (5-15), (5-19b), and (5-21) is

$$\sum_{k=0}^{N-1} D(f - \overline{f}_k) D(\overline{f}_k - f') = \sum_{k-0}^{N-1} D\star(\overline{f}_k - f) D(\overline{f}_k - f')$$

$$= D(f - f') \; , \tag{5-22a}$$

a special case of which is

$$\sum_{k=0}^{N-1} \left| D(\overline{f}_k - f) \right|^2 = 1 \; . \tag{5-22b}$$

After multiplying Equation (5-21) by $NT_0 D(f - \overline{f}_j)$, integrating over f and making use of Equation (5-6), we find that

5-8

$$\hat{v}(\overline{f}_j) = NT_0 \int_{-F_0/2}^{F_0/2} \hat{v}(f)D(f - \overline{f}_j)df \; ; \; 0 \le j \le N - 1 \; . \tag{5-23}$$

From the above equations, it also follows that

$$\hat{v}(f) = NT_0 \Theta(f) \int_{-F_0/2}^{F_0/2} \hat{v}(f')D(f' - f)df' \; . \tag{5-24}$$

5.4   SELF-CONJUGATE BASES

For an arbitrary choice of $\overline{f}$ in Equation (5-18), the set of vectors

$$\hat{v}(-\overline{f}_k) = \hat{v}*(\overline{f}_k) \; ; \; 0 \le k \le N - 1 \; , \tag{5-25}$$

differs from the original set. This is not of any concern as long as we are considering arbitrary complex vectors, but, when our interest is in real vectors, it can be very desirable to have the set of basis vectors equal to its complex conjugate set; that is, for the basis to be self-conjugate. One way to guarantee this property is to choose $\overline{f} = 0$, in which case the set of basis frequencies is given in Equation (5-17) -- this is the basis almost always used in the literature. In this case there are two real vectors,

$$\hat{v}(f_0) = \hat{v}*(f_0) \; ,$$
$$\hat{v}(f_{N/2}) = \hat{v}*(f_{N/2}) \; , \tag{5-26a}$$

N/2 - 1 positive frequency vectors,

$$\hat{v}(f_k) \; ; \; 1 \leq k \leq \frac{N}{2} - 1 \; , \tag{5-26b}$$

and $N/2 - 1$ negative frequency vectors

$$\hat{v}(f_k) = \hat{v}(-f_{N-k}) = \hat{v}*(f_{N-k}) \; ; \; \frac{N}{2} + 1 \leq k \leq N - 1 \; . \tag{5-26c}$$

Equation (5-20) can, in this case, be written

$$\hat{v}(f_k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{e}(n) e^{j2\pi nk/N} \; ; \; 0 \leq k \leq N - 1 \; , \tag{5-27a}$$

and

$$\hat{e}(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{v}(f_k) e^{-j2\pi nk/N} \; ; \; 0 \leq n \leq N - 1 \; . \tag{5-27b}$$

Another choice of $\bar{f}$, such that the basis vectors and the complex conjugate set are identical, is

$$\bar{f} = \frac{F_0}{2N} \; . \tag{5-28}$$

In this case, the basis frequencies

$$\overset{\circ}{f}_k = \begin{cases} (2k + 1) \dfrac{F_0}{2N} \; ; \; 0 \leq k \leq \dfrac{N}{2} - 1 \; , \\[3mm] -(2N - 2k - 1) \dfrac{F_0}{2N} \; ; \; \dfrac{N}{2} \leq k \leq N - 1 \; , \end{cases} \tag{5-29a}$$

satisfy

$$\overset{o}{f}_{N-k-1} = - \overset{o}{f}_k \; ; \; 0 \leq k \leq \frac{N}{2} - 1 \; . \tag{5-29b}$$

The basis now consists of the N/2 positive frequency vectors

$$\hat{v}(\overset{o}{f}_k) \; ; \; 0 \leq k \leq \frac{N}{2} - 1 \; , \tag{5-30a}$$

and the N/2 negative frequency ones

$$\hat{v}(\overset{o}{f}_k) = \hat{v}(-\overset{o}{f}_{N-k-1}) = \hat{v}*(\overset{o}{f}_{N-k-1}) \; ; \; \frac{N}{2} \leq k \leq N - 1 \; . \tag{5-30b}$$

Equation (5-20) can now be written

$$\hat{v}(\overset{o}{f}_k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{e}(n) e^{j\pi n(2k+1)/N} \; ; \; 0 \leq k \leq N - 1 \; , \tag{5-31a}$$

and

$$\hat{e}(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{v}(\overset{o}{f}_k) e^{-j\pi n(2k+1)/N} \; ; \; 0 \leq n \leq N - 1 \; . \tag{5-31b}$$

SECTION 6

FREQUENCY REPRESENTATION OF A FINITE TIME SERIES

6.1 BASIS FREQUENCY FOURIER AMPLITUDES

It has been shown that the frequency-domain basis vector $\varepsilon(f)$ in C^∞ has the image

$$\hat{\varepsilon}(f) = \sqrt{N}\,\hat{v}(f)$$

in the vector space C^N to which actual measurements correspond, where $\hat{v}(f)$ is a unit vector, and that one can choose N of the vectors $\hat{v}(f)$ for an orthonormal basis. This section discusses the frequency-domain representation of an arbitrary vector in C^N.

When Equations (3-11b) and (5-20b) are substituted into Equation (4-10) for the mapping operator $\tilde{M}$ from C^∞ to C^N, the result is

$$\tilde{M} = \sqrt{N}\,T_0 \sum_{k=0}^{N-1} \int_{-F_0/2}^{F_0/2} \hat{v}(\overline{f}_k)D(\overline{f}_k - f)\varepsilon^\star(f)df \ . \tag{6-1}$$

When this operator is applied to a vector x in C^∞ expressed in the frequency representation, as in Equation (3-14a), we obtain

$$\hat{x} = \sum_{k=0}^{N-1} \hat{v}(\overline{f}_k)\hat{X}(\overline{f}_k) \ , \tag{6-2a}$$

where the discrete Fourier amplitudes $\hat{X}(\overline{f}_k)$ are given by

$$\hat{X}(\overline{f}_k) = \sqrt{N} \int_{-F_0/2}^{F_0/2} D(\overline{f}_k - f)X(f)df \; ; \; 0 \leq k \leq N - 1 \; . \qquad (6\text{-}2b)$$

Thus, just as Equation (4-7a) gives the time-domain representation of the vector $\hat{x}$ in C^N, Equation (6-2) gives the frequency-domain representation using one of the frequency bases given by Equations (5-17) and (5-19). From Equation (6-2a),

$$\hat{X}(\overline{f}_k) = \mathbf{\Psi}*(\overline{f}_k) \cdot \hat{x} \; ; \; 0 \leq k \leq N - 1 \; . \qquad (6\text{-}3)$$

Note that the scalar product in C^N, which was introduced in Equation (4-3), can also be written

$$(\hat{y}, \hat{x})_N = \sum_{k=0}^{N-1} \hat{Y}*(\overline{f}_k)\hat{X}(\overline{f}_k) \; . \qquad (6\text{-}4)$$

When Equation (5-20b) is inserted into the right-hand side of Equation (4-7a), we also find that

$$\hat{X}(\overline{f}_k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-j2\pi n T_0 \overline{f}_k} \; ; \; 0 \leq k \leq N - 1 \; , \qquad (6\text{-}5a)$$

which has the inverse

$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}(\overline{f}_k) e^{j2\pi n T_0 \overline{f}_k} \; ; \; 0 \leq n \leq N - 1 \; . \qquad (6\text{-}5b)$$

The two cases of special interest are for the basis frequencies f_k in Equation (5-17) and for the basis frequencies $\mathring{f}_k$ in Equation (5-29). In the former case

$$\hat{X}(f_k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N} \; ; \; 0 \leq k \leq N - 1 \; ,$$

(6-6a)

and

$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}(f_k) e^{j2\pi nk/N} \; ; \; 0 \leq n \leq N - 1 \; ,$$

(6-6b)

while in the latter case we have

$$\hat{X}(\mathring{f}_k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N=1} x_n e^{-j\pi n(2k+1)/N} \; ; \; 0 \leq k \leq N - 1 \; ,$$

(6-7a)

and

$$x_n = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{X}(\mathring{f}_k) e^{j\pi n(2k+1)/N} \; ; \; 0 \leq n \leq N - 1 \; .$$

(6-7b)

If we are considering a real time series g_k and its vector $\hat{g}$, instead of Equation (6-2) we have

$$\hat{g} = \sum_{k=0}^{N-1} \mathfrak{v}(\overline{f}_k) \hat{G}(\overline{f}_k) \; ,$$

(6-8a)

6-3

where

$$\hat{G}(\overline{f}_k) = \sqrt{N} \int_{-F_0/2}^{F_0/2} D(\overline{f}_k - f)G(f)df \ ; \ 0 \leq k \leq N - 1 \ . \tag{6-8b}$$

Because of Equations (2-3) and (5-5), we find that

$$\hat{G}*(\overline{f}_k) = G(-\overline{f}_k) \ ; \ 0 \leq k \leq N - 1 \ , \tag{6-9a}$$

so it is convenient to use one of the two sets of basis frequencies that is identical to its complex conjugate set, i.e., either the set in Equation (5-26), in which case $\hat{G}(f_0)$ and $\hat{G}(f_{N/2})$ are real and

$$\hat{G}*(f_k) = G(f_{N-k}) \ ; \ 1 \leq k \leq \frac{N}{2} - 1 \ , \tag{6-9b}$$

or the set in Equation (5-30), in which case

$$\hat{G}*(\mathring{f}_k) = G(\mathring{f}_{N-k-1}) \ ; \ 0 \leq k \leq \frac{N}{2} - 1 \ . \tag{6-9c}$$

Otherwise, if we use an arbitrary choice of basis, each of the amplitudes $G(f_k)$ for $N/2 \leq k \leq N - 1$ is a linear combination of the complex conjugate amplitudes for $0 \leq k \leq N/2 - 1$.

6.2 FOURIER AMPLITUDES FOR ANY FREQUENCY

Let us now return to the case of an arbitrary complex time series. In order to excise narrowband interference, we will need to know the component of $\hat{x}$ in the direction of $\mathring{V}(f)$, i.e.,

$$\hat{X}(f) = \mathbf{\mathscr{v}}*(f) \cdot \hat{x} , \qquad\qquad (6\text{-}10)$$

an expression that has the same form as Equation (6-3) for the basis frequencies. If we write $\hat{x}$ in the time representation as in Equation (4-7a) and use Equation (5-13c) for $\mathscr{v}(f)$, we have

$$\hat{X}(f) = \Theta(f) \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-j2\pi k T_0 f} . \qquad\qquad (6\text{-}11a)$$

On the other hand, if we express $\hat{x}$ in the frequency representation, as in Equation (6-2a) and make use of Equation (5-15), we find that

$$\hat{X}(f) = \Theta(f) \sum_{k=0}^{N-1} D(f - \overline{f}_k) \hat{X}(\overline{f}_k) . \qquad\qquad (6\text{-}11b)$$

Thus $\hat{X}(f)$, which we shall call the finite Fourier amplitude for frequency f, can be obtained from either the time series components x_k using the finite Fourier transform in Equation (6-11a) or from the discrete Fourier amplitudes $\hat{X}(\overline{f}_k)$ for the basis frequencies using Equation (6-11b).

Equation (6-2b) is an expression for the discrete Fourier amplitudes for the basis frequencies in terms of the Fourier amplitudes in C^∞, but there is no way one can solve for the latter. This equation does tell us that the amplitude for any of the basis frequencies is a supposition of both positive and negative frequency amplitudes in C^∞. If it is substituted into the right-hand side of Equation (6-11b) we find that

$$\hat{X}(f) = \sqrt{N} \Theta(f) \int_{-F_0/2}^{F_0/2} D(f-f') X(f') df' \qquad\qquad (6\text{-}12a)$$

$$= NT_0 \, \Theta(f) \int\limits_{-F_0/2}^{F_0/2} D(f-f')\hat{X}(f')df' \; . \tag{6-12b}$$

A special case of the above is

$$\hat{X}(\overline{f}_k) = NT_0 \int\limits_{-F_0/2}^{F_0/2} D(\overline{f}_k - f)\hat{X}(f)df \; . \tag{6-13}$$

When the above equations are substituted into Equation (6-2a), we find, with the help of Equation (5-21) that

$$\hat{x} = \sqrt{N} \int\limits_{-F_0/2}^{F_0/2} \hat{V}(f)X(f)df \tag{6-14a}$$

$$= NT_0 \int\limits_{-F_0/2}^{F_0/2} \hat{V}(f)\hat{X}(f)df \; . \tag{6-14b}$$

6.3 FREQUENCY PROJECTION OPERATORS

The unit operator $\tilde{I}_N$ introduced in Equation (4-23) can be written, in the frequency representation

$$\tilde{I}_N = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{v}(\overline{f}_k) \delta_{kl} \hat{v}*(\overline{f}_l) \tag{6-15a}$$

$$= \sum_{k=0}^{N-1} \hat{v}(\overline{f}_k) \hat{v}*(\overline{f}_k) \ . \tag{6-15b}$$

Another operator of special interest is the projection operator

$$\tilde{P}_N(f) = \hat{v}(f)\hat{v}*(f) \ . \tag{6-16}$$

Given an arbitrary vector $\hat{x}$ in C^N, consider the vector

$$\hat{x}(f) = \tilde{P}_N(f) \cdot \hat{x} \tag{6-17a}$$

$$= \hat{v}(f)[\hat{v}*(f) \cdot \hat{x}] \tag{6-17b}$$

$$= \hat{v}(f)\hat{X}(f) \ , \tag{6-17c}$$

We see that $\hat{x}(f)$ is the projection of the vector $\hat{x}$ in the direction of the unit vector $\hat{v}(f)$ corresponding to the frequency f. Since $\hat{X}(f)$ is the finite Fourier amplitude for frequency f, we shall refer to $\hat{x}(f)$ as the finite Fourier vector for that frequency. Because of Equation (5-21)

$$\hat{x}(f) = \sum_{k=0}^{N-1} \hat{v}(\overline{f}_k)\hat{X}(f;\overline{f}_k) \ , \tag{6-18a}$$

where

$$\hat{X}(f;\overline{f}_k) = D(\overline{f}_k - f)\hat{X}(f); \ 0 \le k \le N - 1 \ . \tag{6-18b}$$

The properties of projection operators are discussed in reference 14.

6.4   POSITIVE AND NEGATIVE FREQUENCY AMPLITUDES

In Equations (3-16) and (3-17), we saw that an arbitrary vector $\mathbf{x}$ in C^∞ can be written as the sum of a positive frequency part $\mathbf{x}(+)$ and a negative frequency part $\mathbf{x}(-)$, where $\mathbf{x}(+)$ and $\mathbf{x}(-)$ are orthogonal. The image of $\mathbf{x}(\pm)$ in C^N is

$$\hat{\mathbf{x}}(\pm) = \tilde{\mathbf{M}} \cdot \mathbf{x}(\pm) \ . \tag{6-19}$$

From Equation (6-2) it follows that

$$\hat{\mathbf{x}}(\pm) = \sum_{k=0}^{N-1} \hat{\mathbf{w}}(\overline{f}_k) \hat{X}(\pm, \overline{f}_k) \ , \tag{6-20}$$

where

$$\hat{X}(+, \overline{f}_k) = \sqrt{N} \int_0^{F_0/2} D(\overline{f}_k - f) X(f) df \ ; \ 0 \leq k \leq N - 1 \ , \tag{6-21a}$$

and

$$\hat{X}(-, \overline{f}_k) = \sqrt{N} \int_0^{F_0/2} D(\overline{f}_k + f) X(-f) df \ ; \ 0 \leq k \leq N - 1 \ . \tag{6-21b}$$

This result resembles Equation (6-2b), except for the restriction on the range of integration. In fact, $\hat{X}(\overline{f}_k)$ is related to the amplitudes above by

$$\dot{X}(\overline{f}_k) = \dot{X}(+,\overline{f}_k) + \dot{X}(-,\overline{f}_k) \ ; \ 0 \leq k \leq N - 1 \ . \tag{6-22}$$

Although only positive frequency amplitudes in $C^\bullet$ contribute to Equation (6-21a) and only negative frequency ones to Equation (6-21b), both $\hat{x}(+)$ and $\hat{x}(-)$ have components along all of the frequency basis vectors, not just the positive frequency basis vectors in the case of $\hat{x}(+)$ and the negative frequency ones in the case of $\hat{x}(-)$. Furthermore, in contrast to Equation (3-18), which tells us that $x(+)$ and $x(-)$ are orthogonal in $C^\bullet$, we have

$$(\hat{x}(-),\hat{x}(+))_N = \hat{x}(-)^* \cdot \hat{x}(+) \tag{6-23a}$$

$$= \sum_{k=0}^{N-1} \left[\hat{X}(-,\overline{f}_k)\right]^* \hat{X}(+,\overline{f}_k) \tag{6-23b}$$

$$= N \int_0^{F_0/2} df \int_{-F_0/2}^{0} df' X^*(f')D(f' - f)X(f) \ . \tag{6-23c}$$

Thus, $\hat{x}(+)$ and $\hat{x}(-)$ are not generally orthogonal, and there is no way we can determine these two vectors from a knowledge of only $\hat{x}$, but when $D(f' - f)$ is small for the range of values for which $X^*(f')$ and $X(f)$ are large in Equation (6-23c), we can determine them approximately.

It is ambiguous to talk about decomposing an arbitrary vector $\hat{x}$ in C^N into a "positive frequency" part and a "negative frequency" part. Suppose we make some choice of basis vectors $\overline{f}_k$ for a particular choice of $\overline{f}$ in Equation (5-18) and consider another choice

$$\overline{f}_k' = \overline{f}' + f_k \ ; \ 0 \leq k \leq N - 1 \ , \tag{6-24a}$$

6-9

where

$$\frac{F_o}{N} > \overline{f}' \geq 0 \; ; \; \overline{f}' \neq f \; . \tag{6-24b}$$

It follows from Equation (6-11b) that

$$\hat{X}(\overline{f}_k') = \sum_{n=0}^{N-1} D(\overline{f}_k' - \overline{f}_n)\hat{X}(\overline{f}_n) \; ; \; 0 \leq k \leq N - 1 \; . \tag{6-25}$$

We see that each amplitude $\hat{X}(\overline{f}_k')$ is a linear combination of all of the amplitudes $\hat{X}(\overline{f}_n)$. There is no decomposition of the two sets of amplitudes into subsets of "positive frequency" and "negative frequency" amplitudes such that the primed amplitudes of a given frequency sign are a linear combination of only unprimed amplitudes having the same frequency sign.

6.5 SINE WAVES

To conclude this section, let us look at the effect that the projection into a finite time series has on the vector $\boldsymbol{\omega_a}$ in Equation (3-22) representing a sinusoid with a frequency F_a. The projection of $\boldsymbol{\omega_a}$ onto C^N is

$$\hat{\boldsymbol{\omega}}_a = \tilde{M} \cdot \boldsymbol{\omega}_a \tag{6-26a}$$

$$= A_a \sum_{k=0}^{N=1} \hat{e}(k)\cos(2\pi F_a k T_0 - \phi_a) \tag{6-26b}$$

$$= \sqrt{N}\frac{A_a}{2}\left[e^{-j\phi_a}\mathcal{V}(F_a) + e^{j\phi_a}\mathcal{V}^*(F_a)\right] \; , \tag{6-26c}$$

6-10

and the scalar product of two such sine waves is

$$(\hat{\omega}_b, \hat{\omega}_a)_N = \hat{\omega}_b^* \cdot \hat{\omega}_a \tag{6-27a}$$

$$= N \frac{A_a A_b}{2} \left\{ \cos\left[(N-1)\pi T_0 (F_a - F_b) - (\phi_a - \phi_b)\right] D^\circ(F_a - F_b) \right.$$

$$\left. + \cos\left[(N-1)\pi T_0 (F_a + F_b) - (\phi_a + \phi_b)\right] D^\circ(F_a + F_b) \right\} . \tag{6-27b}$$

Consequently, the length of $\hat{\omega}_a$ is

$$||\hat{\omega}_a|| = \sqrt{\frac{N}{2}} A_a \left\{ 1 + \cos\left[2(N-1)\pi T_0 F_a - 2\phi_a\right] D^\circ(2F_a) \right\}^{1/2} \tag{6-28}$$

The length of $\hat{\omega}_a$ therefore depends on both the frequency F_a and the phase ϕ_a, except when $D^\circ(2F_a) = 0$.

SECTION 7

INTERFERENCE EXCISION IN A COMPLEX TIME SERIES

7.1 NARROWBAND INTERFERER IN AN INFINITE TIME SERIES

Before the excision of an arbitrary number of narrowband interferers is discussed, the problem will be simplified as much as possible by allowing only a single interferer. Since a single interference band for a real time series actually means two interfering bands, one in the positive frequency region and another in the negative frequency one, this section and the next one will consider a single narrowband interferer in a complex time series.

The finite time series in which we are interested is taken from the infinite band-limited series

$$x_k = s_k + b_k \; ; \; k = 0, \pm 1, \pm 2, \ldots, \tag{7-1}$$

where

$$s_k \; ; \; k = 0, \pm 1, \pm 2, \ldots,$$

is the signal and

$$b_k \; ; \; k = 0, \pm 1, \pm 2, \ldots,$$

is the narrowband interference. For the purposes of this paper, the signal is a pseudonoise sequence. The random noise with which one is usually concerned can be included in the signal and will not be discussed. The Fourier transform of this time series has the form

$$X(f) = S(f) + B(f) \; , \qquad\qquad\qquad (7\text{-}2)$$

where $X(f)$ is expressed in terms of the series x_k by means of the discrete-time Fourier transform in Equation (2-23a) and where $S(f)$ and $B(f)$ are similarly related to s_k and b_k, respectively.

The spectrum of the signal is spread throughout the frequency range

$$\frac{F_0}{2} \geq f \geq -\frac{F_0}{2}$$

and is relatively flat. We assume that $B(f)$, on the other hand, is essentially zero outside a narrow frequency band

$$F_2 \geq f \geq F_1 \; , \qquad\qquad\qquad (7\text{-}3a)$$

where

$$\frac{F_0}{2} > F_2 > F_1 > 0 \qquad\qquad\qquad (7\text{-}3b)$$

and

$$F_2 - F_1 \ll \frac{F_0}{2} \quad . \qquad\qquad\qquad (7\text{-}3c)$$

That is,

$$B(f) = \begin{cases} 0 & ; \; 0 \leq f < F_1 \; , \\ \bar{B}(f) & ; \; F_1 \leq f \leq F_2 \; , \\ 0 & ; \; F_2 < f \; , \end{cases} \qquad (7\text{-}4a)$$

and

$$B(f) = 0 \; ; \; f \leq 0 \; . \tag{7-4b}$$

7.2 NARROWBAND INTERFERER IN A FINITE TIME SERIES

In practice, of course, we measure only the finite time series

$$x_k = s_k + b_k \; ; \; 0 \leq k \leq N - 1 \; , \tag{7-5}$$

and we know only x_k, not s_k and b_k separately, for this time interval. After choosing one of the basis frequency sets allowed by Subsection 5.3, we have, as a result of a finite Fourier transform,

$$\hat{X}(\overline{f}_k) = \hat{S}(\overline{f}_k) + \hat{B}(\overline{f}_k) \; ; \; 0 \leq k \leq N - 1 \; . \tag{7-6}$$

Here $\hat{S}(\overline{f}_k)$ is the Fourier transform of the finite series s_k and $\hat{B}(\overline{f}_k)$ is that of the finite series b_k, but, of course, we can only calculate $\hat{X}(\overline{f}_k)$ and not $\hat{S}(\overline{f}_k)$ and $\hat{B}(\overline{f}_k)$ separately. The corresponding vectors in C^N are

$$\hat{x} = \hat{s} + \hat{b} \; , \tag{7-7}$$

where the time and frequency representations of $\hat{x}$ are given by Equations (4-7a) and (6-2a), respectively, and where, similarly,

$$\hat{s} = \sum_{k=0}^{N-1} \hat{e}(k)s_k = \sum_{k=0}^{N-1} \hat{v}(k)\hat{S}(\overline{f}_k) \; , \tag{7-8a}$$

$$\hat{b} = \sum_{k=0}^{N-1} \hat{e}(k)b_k = \sum_{k=0}^{N-1} \hat{v}(k)\hat{B}(\overline{f}_k) \; . \tag{7-8b}$$

The discrete Fourier amplitudes $\hat{X}(\overline{f}_k)$ in C^N are related to the continuous amplitude $X(f)$ in C^∞ by Equation (6-2b). Likewise,

$$\hat{S}(\overline{f}_k) = \sqrt{N} \int_{-F_0/2}^{F_0/2} D(\overline{f}_k - f)S(f)df \; ; \; 0 \leq k \leq N - 1 , \qquad (7-9a)$$

but, because of Equation (7-4),

$$\hat{B}(\overline{f}_k) = \sqrt{N} \int_{F_1}^{F_2} D(\overline{f}_k - f)B(f)df \; ; \; 0 \leq k \leq N - 1 . \qquad (7-9b)$$

A consequence of this last equation is that the narrow interference band defined by Equations (7-3) and (7-4) contributes to all of the amplitudes $\hat{B}(\overline{f}_k)$, not just to the amplitudes for those basis frequencies that lie in the band. In fact, even though the integral in Equation (7-9b) is over a small positive-frequency range, it results in a contribution not only to the "positive frequency" amplitudes $\overline{f}_k$ such that

$$0 \leq \overline{f}_k \leq \overline{f}_{(N/2)-1} \; ,$$

but also to the "negative frequency" amplitudes. All of this is a result of the finite width of the central lobe of $D(f)$ and of its sidelobes.

7.3 THE EXCISION PROBLEM

Let us suppose that some noise threshold N_o has been established such that

$$N_o >> |\hat{S}(\overline{f}_k)| \; ; \; 0 \le k \le N - 1 \; , \tag{7-10}$$

and that we wish to excise interference that exceeds this threshold. It is not necessary to be concerned about narrowband interference whose peak amplitude is comparable in magnitude to that of the signal, since its power will be small. We assume that there exists some narrow band of basis frequencies $\bar{f}_k$:

$$\overline{f}_K \ge \overline{f}_k \ge \overline{f}_J \; , \tag{7-11a}$$

where

$$\frac{F_0}{2} > \overline{f}_K \ge \overline{f}_J > 0 \tag{7-11b}$$

and

$$\overline{f}_k - \overline{f}_J << \frac{F_0}{2} \; , \tag{7-11c}$$

such that

$$|\hat{B}(\overline{f}_k)| \ge N_o \; ; \; J \le k \le K \; . \tag{7-11d}$$

Although the data enables the calculation of only the amplitudes $\hat{X}(\bar{f}_k)$ in Equation (7-6), and neither $\hat{S}(\bar{f}_k)$ nor $\hat{B}(\bar{f}_k)$, within this band,

$$\hat{X}(\overline{f}_k) \simeq \hat{B}(\overline{f}_k) \; ; \; J \le k \le K \; . \tag{7-12}$$

We cannot excise the interference simply by putting

$$\hat{X}(\overline{f}_k) = 0 \ ; \ J \leq k \leq K \ , \tag{7-13}$$

because a great deal of power in the narrowband may be distributed among basis frequencies outside the range in Equation (7-11a). Therefore, there may be a number of basis frequencies outside that range for which $\hat{B}(\overline{f}_k)$ is not negligible compared to $\hat{S}(\overline{f}_k)$. A method is needed to account for the contributions that non-basis frequencies in the band make to basis frequencies outside of it through the integral in Equation (7-9b).

We know from the discussion in Sections 5 and 6 that any choice of basis frequencies allowed by Equations (5-17) and (5-18) is appropriate. Clearly, the rules employed to design a filter to suppress the interference and the results of applying this filter should not depend on the basis chosen. If the rule in Equation (7-13) is applied for a particular basis, it cannot be valid for any other basis.

The rule in Equation (7-11d), strictly speaking, is basis dependent. This is because, even if Equation (7-11d) is not satisfied for any value of k, it is possible for some amplitudes $\hat{B}(f)$ for frequencies lying between the basis frequencies to exceed the noise threshold -- any such frequency is a possible basis frequency in some other basis. It will generally be sufficient to test the basis frequency amplitudes only, but, if one wishes to be more particular, the amplitudes for the frequencies halfway between the basis frequencies can also be checked.

7.4 LEAST SQUARES FORMALISM FOR THE INTERFERENCE SUBSPACE

We will seek the subspace of the vector space C^N that best fits the narrowband interference, and we will then delete this subspace to suppress the interference. In this subsection a least squares formalism for finding

the subspace is given, and a matrix solution is found in the next subsection. Finally, in Subsection 7.6 we will show how to excise the interference by projecting the data vector onto the subspace orthogonal to it.

We know from Section 6 that one can easily evaluate the Fourier amplitude $\hat{X}(f)$ for non-basis frequencies. We can, by such calculations, find some range of frequencies such that, in place of Equations (7-11) and (7-12) we have

$$\left| \hat{X}(f) \right| \simeq \left| \hat{B}(f) \right| \geq N_o \; ; \; \overline{F}_2 \geq f \geq \overline{F}_1 \; , \tag{7-14a}$$

where

$$\overline{f}_{K+1} > \overline{F}_2 \geq \overline{f}_K \tag{7-14b}$$

and

$$\overline{f}_J \geq \overline{F}_1 > \overline{f}_{J-1} \; . \tag{7-14c}$$

In practice, we need only an approximate knowledge of F_2 and F_1. (As a matter of fact, in order to be certain that the interference band is properly suppressed, one may wish to include some frequencies for which $\left| \hat{X}(f) \right|$, although very large, does not satisfy Equation (7-14a) -- then $\overline{F}_2$ may be larger than $\overline{f}_{K+1}$ and $\overline{F}_1$ smaller than $\overline{f}_{J-1}$.)

Let us now choose some number, say L, of frequencies in the range in Equation (7-14), where

$$N \gg L > K - J + 1 \; , \tag{7-15}$$

and let us label these frequencies as follows:

$$\phi_p \; ; \; p = 1 \; , \; . \; . \; . \; , \; L \; , \tag{7-16a}$$

where

$$\overline{F}_2 \geq \phi_L > \phi_{L-1} > \; . \; . \; . \; > \phi_2 > \phi_1 \geq \overline{F}_1 \; . \tag{7-16b}$$

The exact number L and the correct way to choose these frequencies will not be decided in this paper. It is argued in Appendix A that L should be at least four times larger than K - J, i.e.,

$$N \gg L \gtrsim 4(K - J) + 1 \; . \tag{7-17}$$

In fact, the frequencies may even be selected so that they are equally spaced, by dividing the interval between the frequencies F_1 and F_2 into L - 1 equal parts. But the formalism to follow does not depend on the specific method by which we choose the frequencies in Equation (7-16), and we will leave it to practical experience to make the final determination.

Using either of Equations (6-11), we can now calculate

$$\hat{X}(\phi_p) \simeq \hat{B}(\phi_p) \; ; \; 1 \leq p \leq L \; . \tag{7-18}$$

According to Equation (6-17), the Fourier vectors in C^N for these frequencies are

$$\hat{x}(\phi_p) = \mathcal{V}(\phi_p)\hat{X}(\phi_p) \simeq \mathcal{V}(\phi_p)\hat{B}(\phi_p) \; ; \; 1 \leq p \leq L \; . \tag{7-19}$$

The vector $\hat{x}(\phi_p)$ is the projection of the data vector $\hat{x}$ onto the direction of the unit vector $\mathcal{V}(\phi_p)$.

Let us consider an arbitrary complex unit vector $\hat{u}$ and the projection operators [14]

$$\tilde{P}_N(\hat{u}) = \hat{u}\hat{u}^* \ , \tag{7-20a}$$

$$\tilde{P}_N^0(\hat{u}) = \tilde{I}_N - \tilde{P}_N(\hat{u}) \ . \tag{7-20b}$$

The vector

$$\tilde{P}_N(\hat{u}) \cdot \hat{x}(\phi_p) = \hat{u}[\hat{u}^* \cdot \hat{x}(\phi_p)] \tag{7-21a}$$

is the projection of $\hat{x}(\phi_p)$ onto the direction of $\hat{u}$, and the vector

$$\tilde{P}_N^0(\hat{u}) \cdot \hat{x}(\phi_p) = \hat{x}(\phi_p) - \hat{u}[\hat{u}^* \cdot \hat{x}(\phi_p)] \tag{7-21b}$$

is the projection of $\hat{x}(\phi_p)$ onto the subspace of C^N orthogonal to $\hat{u}$.

Let us now ask: What is the vector $\dot{u}$ such that the sum of the squared distances

$$\Lambda^0 = \sum_{p=1}^{L} \mu_p \left| \tilde{P}_N^0(\hat{u}) \cdot \hat{x}(\phi_p) \right|^2 \tag{7-22a}$$

is a minimum? Here μ_p is a weighting factor that will be briefly discussed below. The vector $\hat{u}$ that answers this question is the best fit in the least squares sense to the subspace defined by the interference band frequency vectors in Equation (7-19) [15]. This is the same as asking for the vector $\hat{u}$ such that the sum of the squared distances

$$\Lambda = \sum_{p=1}^{L} \mu_p \left| \bar{P}_N(\hat{u}) \cdot \hat{x}(\phi_p) \right|^2 = \sum_{p=1}^{L} \mu_p \left| \hat{u}\star \cdot \hat{x}(\phi_p) \right|^2 \tag{7-22b}$$

is a maximum.

Equation (7-22b) should be regarded as an approximation to the integral

$$\Lambda = \int_{\bar{F}_1}^{\bar{F}_2} df \left| \hat{u}\star \cdot \hat{x}(f) \right|^2 \tag{7-23}$$

over the interference band, where the frequencies $\bar{F}_1$ and $\bar{F}_2$ were defined in Equation (7-14). If the frequencies ϕ_p are chosen to be equally spaced, with

$$\phi_1 = \bar{F}_1 \ , \tag{7-24a}$$

$$\phi_L = \bar{F}_2 \ , \tag{7-24b}$$

the simplest possible choice for the weights μ_p in Equation (7-22) is

$$\mu_p = \begin{cases} (\bar{F}_2 - \bar{F}_1)/(L - 1) \ ; \ 1 \leq p \leq L - 1 \ , \\ 0 \qquad\qquad ; \ p = L \ . \end{cases} \tag{7-24c}$$

If one employs some other numerical integration scheme, with or without uniform intervals between the ϕ_p's, then the appropriate weighting factors $\mu_p > 0$ should be employed [16]. The integral in Equation (7-23) is straightforward but tedious to carry out, and the result is given in Appendix A, which also contains a discussion on the lower limit for L in

Equation (7-17). Our reason for using an approximation instead of the exact integral is that this allows us to greatly reduce the size of the matrix whose largest eigenvalues we must find, as will be shown in Section 8.

7.5 SOLUTION OF THE LEAST SQUARES PROBLEM

According to Equation (6-18), we can expand each of the vectors $\hat{x}(\phi_p)$ in terms of its components along the basis frequencies:

$$\hat{x}(\phi_p) = \sum_{k=0}^{N-1} \mathfrak{v}(\overline{f}_k)\dot{X}(\phi_p,\overline{f}_k) \; ; \; 1 \leq p \leq L \; , \tag{7-25a}$$

where

$$\dot{X}(\phi_p,\overline{f}_k) = D(\overline{f}_k - \phi_p)\hat{X}(\phi_p) \; ; \; 0 \leq k \leq N - 1 \; ; \; 1 \leq p \leq L \; . \tag{7-25b}$$

We can also express any unit vector $\hat{u}$ in terms of its components along the basis frequencies:

$$\hat{u} = \sum_{k=0}^{N-1} \mathfrak{v}(\overline{f}_k)\hat{U}(\overline{f}_k) \; . \tag{7-26a}$$

Thus, in Equation (7-22b),

$$\Lambda = \sum_{p=1}^{L} \sum_{k=0}^{N-1} \mu_p \left| \hat{U}^\star(\overline{f}_k)\hat{X}(\phi_p,\overline{f}_k) \right|^2 \; . \tag{7-26b}$$

It is possible to write the last equation in the form

$$\Lambda = \tilde{U}^+ \tilde{K} \tilde{U} \ , \tag{7-27}$$

where $\tilde{K}$ is an $N \times N$ matrix having the elements

$$K_{k1} = \sum_{p=1}^{L} \mu_p \hat{X}(\phi_p, \overline{f}_k) \hat{X}*(\phi_p, \overline{f}_1) \tag{7-28a}$$

$$= \sum_{p=1}^{L} \mu_p D(\overline{f}_k - \phi_p) |\hat{X}(\phi_p)|^2 D(\phi_p - \overline{f}_1) \ ; \tag{}$$

$$0 \leq k, \ 1 \leq N-1 \ . \tag{7-28b}$$

Here $\tilde{U}$ is the column matrix

$$\tilde{U} = \begin{bmatrix} \hat{U}(\overline{f}_0) \\ \hat{U}(\overline{f}_1) \\ . \\ . \\ . \\ \hat{U}(\overline{f}_{N-1}) \end{bmatrix} \ , \tag{7-29a}$$

and the + symbol, when used as a superscript on any matrix, indicates the Hermitian conjugate or adjoint matrix, i.e., the complex conjugate of its transpose. Thus,

$$\tilde{U}^+ = [\hat{U}*(\overline{f}_0) \ \hat{U}*(\overline{f}_1) \ . \ . \ . \ \hat{U}*(\overline{f}_{N-1})] \ . \tag{7-29b}$$

From Equation (7-28a), we see that the elements of $\tilde{K}$ satisfy

$$\tilde{K}_{kl} = \tilde{K}_{lk}^{\star} \; ; \; 0 \leq k, \; l \leq N - 1 \; , \tag{7-30a}$$

so $\tilde{K}$ is Hermitian, i.e.,

$$\tilde{K} = \tilde{K}^{+} \; . \tag{7-30b}$$

Since $\tilde{K}$ is Hermitian, all of its eigenvalues

$$\lambda_{\alpha} \; ; \; 1 \leq \alpha \leq N \; , \tag{7-31a}$$

are real [13,14]. Let us arrange them in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdot \cdot \cdot \geq \lambda_N \; . \tag{7-31b}$$

The corresponding eigenvectors

$$\tilde{U}^{\alpha} = \begin{bmatrix} \hat{U}^{\alpha}(\overline{f}_0) \\ \hat{U}^{\alpha}(\overline{f}_1) \\ . \\ . \\ . \\ \hat{U}^{\alpha}(\overline{f}_{N-1}) \end{bmatrix} \; ; \; 1 \leq \alpha \leq N \; , \tag{7-32a}$$

satisfy

$$\tilde{K}\tilde{U}^{\alpha} = \lambda_{\alpha}\tilde{U}^{\alpha} \; ; \; 1 \leq \alpha \leq N \; . \tag{7-32b}$$

Eigenvectors corresponding to different eigenvalues are always orthogonal, and for equal eigenvalues they can always be constructed so that they have this property:

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$$(\tilde{U}^{\alpha})^{+}\tilde{U}^{\beta} = \delta_{\alpha\beta} \; ; \; 1 \leq \alpha, \beta \leq N \; . \tag{7-32c}$$

Each of these eigenvectors corresponds to a vector in C^N, i.e.,

$$\hat{u}^{\alpha} = \sum_{k=0}^{N-1} \mathfrak{V}(\bar{f}_k) \hat{U}^{\alpha}(\bar{f}_k) \; ; \; 1 \leq \alpha \leq N \; , \tag{7-33a}$$

and, because of Equation (7-32c),

$$(\hat{u}^{\alpha})^{\star} \cdot \hat{u}^{\beta} = \delta_{\alpha\beta} \; ; \; 1 \leq \alpha, \beta \leq N \; . \tag{7-33b}$$

The column matrix $\tilde{U}$ in Equation (7-29) and the corresponding vector $\hat{u}$ in Equation (7-26a) may now be written

$$\tilde{U} = \sum_{\alpha=1}^{N} \kappa_{\alpha} \tilde{U}^{\alpha} \; , \tag{7-34a}$$

$$\hat{u} = \sum_{\alpha=1}^{N} \kappa_{\alpha} \hat{u}^{\alpha} \; , \tag{7-34b}$$

where the coefficients

$$\kappa_{\alpha} = (\tilde{U}^{\alpha})^{+}\tilde{U} = (\hat{u}^{\alpha})^{\star} \cdot \hat{u} \; ; \; 1 \leq \alpha \leq N \; , \tag{7-35a}$$

satisfy

$$\sum_{\alpha=1}^{N} |\kappa_{\alpha}|^2 = 1 \; . \tag{7-35b}$$

When Equation (7-34a) is substituted into Equation (7-27) and use is made of Equation (7-32), one finds that

$$\Lambda = \sum_{\alpha=1}^{N} |\kappa_\alpha|^2 \lambda_\alpha \; . \tag{7-36a}$$

From the definition of Λ in Equation (7-22b), it cannot be negative; therefore, the N eigenvalues in Equation (7-31) satisfy

$$\lambda_\alpha \geq 0 \; ; \; 1 \leq \alpha \leq N \; . \tag{7-36b}$$

Examination of Equation (7-28) shows that $\tilde{K}$ is a singular matrix and that only the first L eigenvalues are nonzero. Therefore,

$$\lambda_\alpha = 0 \; ; \; L + 1 \leq \alpha \leq N \; , \tag{7-36c}$$

where we recall that L, the number of frequencies chosen in the inter-ference band, satisfies equation (7-17). Equations (7-35b) and (7-36a) give us the answer to the question asked in connection with Equation (7-22a): Λ^0 has its minimum value when $\hat{u}$ is equal to $\hat{u}^1$, the unit vector corresponding to the largest eigenvalue, λ_1, of $\tilde{K}$.

The eigenvalue problem can actually be simplified, because it is easy to show that $\tilde{K}$ is equivalent to a real symmetric matrix. This will not be discussed here because we are going to transform the problem into one in-volving an L $\times$ L matrix in the next section.

7.6 PROJECTION ONTO THE ORTHOGONAL SUBSPACE

It has just been found that Λ is maximized when $\hat{u}$ is equal to the unit vector $\hat{u}^1$ corresponding to the largest eigenvalue of the matrix $\tilde{K}$. This choice of $\hat{u}$ simultaneously minimizes Λ^0 in Equation (7-22a). Let us recall that Λ^0 is the sum of the squared projections of the vectors $\hat{x}(\phi_p)$ in the narrowband onto the subspace of C^N orthogonal to $\hat{u}$. As a result of this latter projection, Equation (7-21b) says that we are left with the vectors

$$\tilde{P}_N^0(\hat{u}^1) \cdot \hat{x}(\phi_p) = \hat{x}(\phi_p) - \hat{u}^1[(\hat{u}^1)^* \cdot \hat{x}(\phi_p)] \; ; \; 1 \leq p \leq L \; . \tag{7-37}$$

Suppose one now asks for the vector $\hat{u}$ such that the sum of the squared absolute values of the projections of the vectors in Equation (7-37) onto the subspace of C^N orthogonal to $\hat{u}$ is a minimum -- the answer is $\hat{u}^2$, the eigenvector corresponding to the second largest eigenvalue of $\tilde{K}$.

Let us now consider the projection operator

$$\tilde{P}_N = \sum_{\alpha=1}^{M} \tilde{P}_N(\hat{u}^\alpha) = \sum_{\alpha=1}^{M} \hat{u}^\alpha (\hat{u}^\alpha)^* \; , \tag{7-38}$$

which is the sum of the projection operators defined by Equation (7-20a) for the eigenvectors belonging to the M largest eigenvalues of $\tilde{K}$, where M is some number that is less than L, the number of nonzero eigenvalues. Let C_0^M be the M-dimensional subspace of C^N spanned by the M vectors

$$\hat{u}^\alpha \; ; \; 1 \leq \alpha \leq M \; ,$$

i.e., an arbitrary vector in C_0^M can be written as a linear combination of these vectors [14]. We see that C_0^M is the M-dimensional subspace of C^N

that best fits in the least squares sense the L frequency vectors $\hat{x}(\phi_p)$ in the interference band. The vector

$$\tilde{P}_N \cdot \hat{x} = \sum_{\alpha=1}^{M} \hat{u}^\alpha [(\hat{u}^\alpha)^* \cdot \hat{x}] \qquad (7\text{-}39)$$

is the projection of the data vector $\hat{x}$ in Equation (7-7) onto C_0^M. The method by which C_0^M has been constructed means that it will include most of the contributions that the sidelobes from the frequencies ϕ_p make to the basis frequency Fourier amplitudes outside the band.

Let us define $\bar{C}_0^{N-M}$ to be the subspace of C^N that is orthogonal to C_0^M. Thus, the operator

$$\tilde{P}_N^0 = \bar{I} - \tilde{P}_N \qquad (7\text{-}40a)$$

is the projection operator onto $\bar{C}_0^{N-M}$; i.e., the vector

$$\hat{x}^0 = \tilde{P}_N^0 \cdot \hat{x} = \hat{x} - \sum_{\alpha=1}^{M} \hat{u}^\alpha [(\hat{u}^\alpha)^* \cdot \hat{x}] \qquad (7\text{-}40b)$$

is the orthogonal projection of the data vector onto $\bar{C}_0^{N-M}$ [14]. This is the subspace of C^N of dimension N-M that minimizes in the least squares sense the effects of the narrowband interference on the data.

The representation of $\hat{x}^0$ in terms of the basis frequencies is

$$\hat{x}^0 = \sum_{k=0}^{N-1} \mathfrak{v}(\bar{f}_k) \hat{X}^0(\bar{f}_k) , \qquad (7\text{-}41a)$$

where the components

$$\hat{X}(\overline{f}_k) = \hat{\mathbf{V}}^*(\overline{f}_k) \cdot \hat{x}^0 \ ; \ 0 \leq k \leq N - 1 \ , \tag{7-41b}$$

are the Fourier amplitudes with the interference from the narrowband suppressed. Because of Equations (6-2a), (7-26a), and (7-40b),

$$\hat{X}^{0'}(\overline{f}_k) = \hat{X}(\overline{f}_k) - \sum_{\alpha=1}^{M} \hat{U}^{\alpha}(\overline{f}_k) \sum_{l=0}^{N-1} [\hat{U}^{\alpha}(\overline{f}_l)]^* \hat{X}(\overline{f}_l) \ ; \ 0 \leq k \leq N - 1 \ . \tag{7-42}$$

What criterion determines M, the dimension of the subspace to be excised? Since, according to Equations (7-11) and (7-12), the original criterion for defining the interference band was

$$|\hat{X}(\overline{f}_k)| \geq N_o \ ; \ J \leq k \leq K \ , \tag{7-43}$$

we should choose M to be the smallest number such that

$$|\hat{X}^0(\overline{f}_k)| \ll N_o \ ; \ J \leq k \leq K \ . \tag{7-44}$$

Strictly speaking, one ought to verify that

$$|\hat{x}^0(\phi_p)| \ll N_o \ ; \ 1 \leq p \leq L \ ,$$

but, in practice, this will almost always be the case when Equation (7-44) is valid. If one is particular, one can check this relation for frequencies that are about halfway between the basis frequencies. (See the remarks in the last paragraph of Subsection 7.3.)

The time series

$$x_k^0 \ , \ 0 \leq k \leq N-1 \ ,$$

with the interference minimized is now found by means of the inverse Fourier transform of the amplitudes $\hat{X}^0(\overline{f}_k)$

$$x_n^0 = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}^0(\overline{f}_k) e^{j2\pi n T_0 \overline{f}_k} \ ; \ 0 \leq n \leq N-1 \ . \tag{7-45}$$

SECTION 8

SIMPLIFIED MATRIX SOLUTION

8.1 TRANSFORMATION TO A SMALLER MATRIX PROBLEM

It has been shown that the problem of excising a single narrowband interferer in a complex time series involves finding the largest eigenvalues and the corresponding eigenvectors of a Hermitian $N \times N$ matrix. But N, the number of time samples in the series of interest, is generally a very large number. In this section it will be demonstrated that the problem can be transformed into one involving a much smaller real symmetric matrix.

According to Equations (7-11) and (7-12), the problem is to excise a narrowband in which $K - J + 1$ basis frequency amplitudes exceed the noise threshold N_o. By some criterion yet to be definitely established, we have chosen L frequencies

$$\phi_p \; ; \; 1 \leq p \leq L \; ,$$

in this band, some of which may coincide with basis frequencies, and have calculated the corresponding Fourier amplitudes and Fourier vectors, which are given in Equations (7-18) and (7-19). As stated in Equation (7-17), L is much smaller than N. The subspace that best fits the Fourier vectors for the L interference frequencies is given by the eigenvectors corresponding to the largest eigenvalues of the $N \times N$ matrix $\tilde{K}$ defined by Equation (7-28). However, $\tilde{K}$ has only L nonzero eigenvalues, so a transformation that changes the problem to that of finding the eigenvalues and eigenvectors of an $L \times L$ real symmetric matrix will now be employed [15].

Because of Equation (5-4), Equation (7-28b) for the elements of $\bar{K}$ can be written

$$K_{k1} = \zeta(\overline{f}_k - \overline{f}_1) \sum_{p=1}^{L} \mu_p D^\circ(\overline{f}_k - \phi_p) |\hat{X}(\phi_p)|^2 D^\circ(\phi_p - \overline{f}_1) , \qquad (8\text{-}1)$$

where

$$\zeta(f) = \exp[-j(N - 1)\pi T_0 f] \qquad (8\text{-}2a)$$

has the property

$$\zeta(f - f') = \zeta(f)\zeta(f') . \qquad (8\text{-}2b)$$

All of the terms in the sum in Equation (8-1) are real.

Let us define a real $N \times L$ matrix $\tilde{X}$; i.e.,

$$\tilde{X} = \left[X_{kp}\right] , \qquad (8\text{-}3a)$$

with the elements

$$X_{kp} = \sqrt{\mu_p} D^\circ(\overline{f}_k - \phi_p) |\hat{X}(\phi_p)| ; \ 0 \le k \le N - 1 ; \ 1 \le p \le L . \qquad (8\text{-}3b)$$

The transpose $\tilde{X}^T$ of $\tilde{X}$ is the $L \times N$ matrix with the elements

$$X^T_{pk} = X_{kp} ; \ 0 \le k \le N - 1 ; \ 1 \le p \le L . \qquad (8\text{-}4)$$

We also introduce the $N \times N$ diagonal matrix $\tilde{Z}$ given by

$$Z_{kl} = \delta_{kl} \zeta(\overline{f}_k) \; ; \; 0 \leq k, l \leq N - 1 \; . \tag{8-5}$$

Therefore,

$$\tilde{K} = \tilde{Z}\tilde{X}\tilde{X}^T\tilde{Z}* \; . \tag{8-6}$$

When one inserts this result for $\tilde{K}$ into Equation (7-32b) and multiplies both sides from the left with $\tilde{X}^T\tilde{Z}*$, the result is

$$(\tilde{X}^T\tilde{X})(\tilde{X}^T\tilde{Z}*\tilde{U}^\alpha) = \lambda_\alpha(\tilde{X}^T\tilde{Z}*\tilde{U}^\alpha) \; ; \; 1 \leq \alpha \leq L \; . \tag{8-7}$$

Let us define the column matrix

$$\tilde{V}^\alpha = \begin{bmatrix} V_1^\alpha \\ V_2^\alpha \\ . \\ . \\ . \\ V_L^\alpha \end{bmatrix} \; ; \; 1 \leq \alpha \leq L \; , \tag{8-8a}$$

by

$$\tilde{V}^\alpha = \frac{1}{\sqrt{\lambda_\alpha}} \tilde{X}^T\tilde{Z}*\tilde{U}^\alpha \; ; \; 1 \leq \alpha \leq L \; . \tag{8-8b}$$

We define the $L \times L$ real symmetric matrix $\tilde{H}$ by

$$\tilde{H} = \tilde{X}^T\tilde{X} \; , \tag{8-9a}$$

so Equation (8-7) becomes

$$\tilde{H}\tilde{V}^\alpha = \lambda_\alpha \tilde{V}^\alpha \; ; \; 1 \leq \alpha \leq L \; . \tag{8-9b}$$

Consequently, the problem of finding the largest eigenvalues of the $N \times N$ Hermitian matrix $\tilde{K}$ has been reduced to that of finding those of the $L \times L$ real symmetric matrix $\tilde{H}$, where $L \ll N$. The eigenvectors $\tilde{V}^\alpha$ can be chosen to be real and, according to Equation (7-32),

$$(\tilde{V}^\alpha)^T \tilde{V}^\beta = \delta_{\alpha\beta} \; ; \; 1 \leq \alpha, \; \beta \leq L \; . \tag{8-9c}$$

8.2 INTERFERENCE EXCISION IN THE NEW FORMALISM

Once one knows the eigenvalues and eigenvectors of $\tilde{H}$, one can easily determine the column matrices $\tilde{U}^\alpha$, whose components are required to minimize the interference, as shown in Section 7. The required relation, which follows from Equations (7-32b) and (8-8), is

$$\tilde{U}^\alpha = \frac{1}{\sqrt{\lambda_\alpha}} \; \tilde{Z}\tilde{X}\tilde{V}^\alpha \; ; \; 1 \leq \alpha \leq L \; . \tag{8-10a}$$

That is,

$$\hat{U}^\alpha(\overline{f}_k) = \frac{1}{\sqrt{\lambda_\alpha}} \; \zeta(\overline{f}_k) \sum_{p=1}^{L} X_{kp} V_p^\alpha \; ; \; 0 \leq k \leq N - 1 \; ; \; 1 \leq \alpha \leq L \; , \tag{8-10b}$$

gives the basis frequency components of the vectors $\hat{u}^\alpha$ that best fit the subspace spanned by the Fourier vectors $\hat{x}(\phi_p)$ in the interference band.

The above allows us to rewrite Equation (7-42) for the Fourier amplitudes $\hat{X}^0(\overline{f}_k)$ with the interference suppressed. Let us define

$$T_{pq} = \sqrt{\mu_p \mu_q} \; |\hat{X}(\phi_p)| |\hat{X}(\phi_q)| \sum_{\alpha=1}^{M} \frac{V_p^\alpha V_q^\alpha}{\lambda_\alpha} \; ; \; 1 \leq p,q \leq L \; . \tag{8-11}$$

8-4

In addition, we introduce

$$\Upsilon_{kq} = \sum_{p=1}^{L} \zeta(\phi_p - \phi_q)D(\overline{f}_k - \phi_p)T_{pq} \tag{8-12a}$$

$$= \zeta(\overline{f}_k - \phi_q)\sum_{p=1}^{L} D^\circ(\overline{f}_k - \phi_p)T_{pq} \; ; \tag{}$$

$$0 \leq k \leq N - 1 \; ; \; 1 \leq q \leq L \; , \tag{8-12b}$$

and

$$\Xi_{kl} = \sum_{p=1}^{L} \sum_{q=1}^{L} \zeta(\phi_p - \phi_q)D(\overline{f}_k - \phi_p)T_{pq}D(\phi_q - \overline{f}_l) \tag{8-13a}$$

$$= \zeta(\overline{f}_k - \overline{f}_l)\sum_{p=1}^{L} \sum_{q=1}^{L} D^\circ(\overline{f}_k - \phi_p)T_{pq}D^\circ(\phi_q - \overline{f}_l) \tag{8-13b}$$

$$= \sum_{q=1}^{L} \Upsilon_{kq}D(\phi_q - \overline{f}_l) \; ; \; 0 \leq k,l \leq N - 1 \; . \tag{8-13c}$$

Then the results can be expressed in either the form

$$\hat{X}^0(\overline{f}_k) = \hat{X}(\overline{f}_k) - \sum_{l=0}^{N-1} \Xi_{kl}\hat{X}(\overline{f}_l) \; ; \; 0 \leq k \leq N - 1 \; , \tag{8-14}$$

or the form

$$\hat{X}^0(\overline{f}_k) = \hat{X}(\overline{f}_k) - \sum_{p=1}^{L} T_{kp}\hat{X}(\phi_p) \; ; \; 0 \leq k \leq N - 1 \; . \qquad (8\text{-}15)$$

Note that the effect on the basis frequency Fourier amplitudes $\hat{X}(\overline{f}_k)$ outside the interference band depends on

$$\left| D(\overline{f}_k - \phi_p) \right| \; ; \; k < J \text{ or } k > K \; ; \; 1 \leq p \leq L \; .$$

The elements of $\tilde{H}$ in Equation (8-9) are

$$H_{pq} = \sum_{k=0}^{N-1} X_{pk}^T X_{kq} = 1 \leq p,q \leq L \; . \qquad (8\text{-}16a)$$

But, because of Equations (5-22a) and (8-3b),

$$H_{pq} = \sqrt{\mu_p \mu_q} \left| \hat{X}(\phi_p) \right| D^{\circ}(\phi_p - \phi_q) \left| \hat{X}(\phi_q) \right| \; ; \; 1 \leq p,q \leq L \; . \qquad (8\text{-}16b)$$

As was discussed in Subsection 7.6, one needs to find only the M largest eigenvalues and the corresponding eigenvectors of $\tilde{H}$, where M is the smallest number required to satisfy Equation (7-44). Note that the Fourier amplitudes $\hat{X}(\phi_p)$, whose magnitudes appear in the matrix elements of both $\tilde{K}$ and $\tilde{H}$, can be expressed as linear combinations of the basis frequency amplitudes by means of Equation (6-11b):

$$\hat{X}(\phi_p) = \sum_{k=0}^{N-1} D(\phi_p - \overline{f}_k)\hat{X}(\overline{f}_k) \; ; \; 1 \leq p \leq L \; . \qquad (8\text{-}17)$$

Here the coefficients $D(\phi_p - \mathfrak{f}_k)$ are subject to the conditions in Equation (5-22b). Since

$$K - J + 1 \ll N$$

of the basis frequency amplitudes exceed the noise threshold, M might be at least as large as this number. Because considerable interference power is distributed among the basis frequencies outside the band, M could possibly be a little larger than $K - J + 1$. On the other hand, if some of our interference band actually includes sidelobes, M may be smaller than this number. In general, we can only conclude that

$$M \sim K - J + 1 \ . \tag{8-18}$$

8.3 TWO EXAMPLES

In this subsection, we will discuss two trivial examples that illustrate two extremes. In the first case the contributions of the non-basis frequencies in the interference band to basis frequencies outside the band are completely ignored, and in the second case these contributions are given too much weight. Since the examples are for illustrative purposes only, the rule about L being at least four times the number of basis frequencies in the band will be ignored and L will be set equal to that number; i.e.,

$$L = K - J + 1 \ .$$

In the first example, we choose the L frequencies in Equation (7-16) to be equal to the basis frequencies in the band; i.e.,

$$\phi_p = \overline{f}_{J-1+p} \; ; \; 1 \leq p \leq K - J + 1 \; . \tag{8-19a}$$

Consequently,

$$\phi_p - \phi_q = \overline{f}_{J-1+p} - \overline{f}_{J-1+q} = f_{J-1+p} - f_{J-1+q}$$

$$= (p-q)\frac{F_0}{N} \; ; \; 1 \leq p,q \leq K - J + 1 \; , \tag{8-19b}$$

where the definition of the basis frequencies in Equations (5-17) and (5-18) has been employed. But, from Equation (5-7b)

$$D^\circ(\phi_p - \phi_q) = D(f_{J-1+p} - f_{J-1+q}) = \delta_{pq} \; ; \; 1 \leq p,q \leq K - J + 1 \; . \tag{8-20}$$

The specific values assigned to the weighting factors μ_p are not important here. Therefore, in Equation (8-16),

$$H_{pq} = \delta_{pq}\mu_p \left| \hat{X}(\overline{f}_{J-1+p}) \right|^2 \; ; \; 1 \leq p,q \leq K - J + 1 \; . \tag{8-21}$$

Thus, $\tilde{H}$ is diagonal and its eigenvalues are the squares of the absolute values of the basis frequency Fourier amplitudes: λ_1 is the largest element of $\tilde{H}$, λ_2 is the second largest, etc., with λ_{K-J+1} the smallest element.

Although an arbitrary ordering of the magnitudes of the elements in Equation (8-21) can easily be handled, in order to simplify things as much as possible (and because it does not affect the final result) we will assume that

$$\lambda_\alpha = \mu_\alpha \left| \hat{X}(\overline{f}_{J-1+\alpha}) \right|^2 \; ; \; 1 \leq \alpha \leq K - J + 1 \; . \tag{8-22}$$

8-8

A consequence of Equations (8-9b) and (8-21) is that the eigenvectors in Equation (8-8a) have the elements

$$V_p^\alpha = \delta_{\alpha p} \; ; \; 1 \leq \alpha , \; p \leq K - J + 1 \; . \tag{8-23}$$

Let us choose the dimension of the subspace to be excised to be

$$M = L = K - J + 1 \; .$$

Then, Equations (8-11) and (8-12) yield the results

$$T_{pq} = \delta_{pq} \; ; \; 1 \leq p,q \leq K - J + 1 \; , \tag{8-24a}$$

$$T_{kq} = \delta_{k , J-1+q} \; ; \; 0 \leq k \leq N - 1 \; ; \; 1 \leq q \leq K - J + 1 \; . \tag{8-24b}$$

Finally, according to Equation (8-15), the Fourier amplitudes with the interference excised are

$$\hat{X}^0(\overline{f}_k) = \begin{cases} \hat{X}(\overline{f}_k) \; ; \; 0 \leq k \leq J - 1 \; , \\ 0 \; ; \; J \leq k \leq K \; , \\ \hat{X}(\overline{f}_k) \; ; \; K + 1 \leq k \leq N - 1 \; , \end{cases} \tag{8-25}$$

a result that does not depend on the particular ordering chosen for the size of the matrix elements in Equation (8-21). Here we have the unacceptable approach discussed in connection with Equation (7-13): the amplitudes for the basis frequencies in the interference band are set equal to zero and there is no effect on the remaining basis frequency amplitudes. Thus, choosing the frequencies in the interference band to be equal to the basis frequencies in the band results in no weight being given to the contributions that non-basis frequencies in the band make to basis frequency amplitudes outside of it.

The second trivial example we will discuss is that in which the frequencies chosen in the interference band are halfway between the basis frequencies. Instead of Equation (8-19a) we now choose

$$\phi_p = \overline{f}_{J-1+p} + \frac{F_0}{2N} \; ; \; 1 \leq p \leq K - J + 1 \; , \tag{8-26}$$

where use has been made of the fact that the interval between basis frequencies is F_0/N. Equations (8-19b) and (8-20) are still valid, but, in place of Equation (8-21), it is found from Equation (8-16) that

$$H_{pq} = \delta_{pq}\mu_p \; |\hat{X}(\overline{f}_{J-1+p} + \frac{F_0}{2N})|^2 \; ; \; 1 \leq p,q \leq K - J + 1 \; . \tag{8-27}$$

Therefore, $\tilde{H}$ is also diagonal in this example.

As we did in Equation (8-22), we shall assume that the magnitudes of the elements of $\tilde{H}$ above are ordered such that

$$\lambda_\alpha = \mu_\alpha \; |\hat{X}(\overline{f}_{J-1+\alpha} + \frac{F_0}{2N})|^2 \; ; \; 1 \leq \alpha \leq K - J + 1 \; . \tag{8-28}$$

This leads to the previous result in Equation (8-23) for the elements of $\tilde{V}^\alpha$. If we once more choose

$$M = L = K - J + 1 \; ,$$

we find from Equations (8-11) and (8-12) that

$$T_{pq} = \delta_{pq} \; ; \; 1 \leq p \; , \; q \leq K - J + 1 \; , \tag{8-29a}$$

$$\Upsilon_{kq} = D(\overline{f}_k - \phi_q) \; ; \; 0 \leq k \leq N - 1 \; ; \; 1 \leq q \leq K - J + 1 \; . \tag{8-29b}$$

8-10

It follows from Equation (8-15) that the Fourier amplitudes with the inter-
ference from the frequencies in Equation (8-26) excised are

$$\hat{X}^0(\overline{f}_k) = \hat{X}(\overline{f}_k) - \sum_{p=1}^{K-J+1} D(\overline{f}_k - \phi_p)\hat{X}(\phi_p) \; ; \; 0 \leq k \leq N - 1 \; . \qquad (8\text{-}30)$$

In the above we have

$$|D(\overline{f}_k - \phi_p)| = \left| \left\{ N \sin\left[\left(J + p - k - \frac{1}{2} \right) \frac{\pi}{N} \right] \right\}^{-1} \right| \; ;$$

$$0 \leq k \leq N - 1 \; ; \; 1 \leq p \leq K - J + 1 \; . \qquad (8\text{-}31)$$

These values are close to the peaks of the sidelobes of the function $D(f)$,
so in this example too much weight is given to the contributions that
non-basis frequencies in the band make to basis frequencies outside of it.

SECTION 9

INTERFERENCE EXCISION IN A REAL TIME SERIES

9.1 NARROWBAND INTERFERENCE IN A REAL TIME SERIES

In the two preceding sections it was shown how one can excise inter-
ference due to a single narrow band from a complex time series. Those
results will now be generalized to a single interference band in a real
time series, which means that there is such a band in both the positive and
the negative frequency regions.

The measured time series is taken from the following real series in
infinite time:

$$g_k = s_k + b_k \; ; \; k = 0, \pm 1, \pm 2, \ldots , \qquad (9\text{-}1)$$

where both the signal s_k and the narrowband interference b_k are real. As
was the case earlier, random noise can, for the purposes of this paper, be
included with the signal, which is a pseudonoise sequence. The Fourier
transform has the form

$$G(f) = S(f) + B(f) , \qquad (9\text{-}2)$$

but now the negative frequency amplitudes for $G(f)$ are related to the posi-
tive frequency ones by Equation (2-3). Similarly,

$$S(-f) = S^\star(f) , \qquad (9\text{-}3a)$$

$$B(-f) = B^\star(f) . \qquad (9\text{-}3b)$$

In the positive frequency region B(f) is still given by Equations (7-3) and (7-4a), but the negative frequency values of B(f) are now given by Equation (9-3b) instead of by Equation (7-4b).

As in Equation (7-5), one actually measures only the finite time series

$$g_k = s_k + b_k \; ; \; 0 \leq k \leq N - 1 \; , \tag{9-4}$$

and knows only g_k, not s_k and b_k separately. Because the time series is real, it is convenient, as discussed in connection with Equation (6-9), to use either the basis frequencies f_k in Equation (5-17) or the basis frequencies $\hat{f}_k$ in Equation (5-29). Since the former basis is the common choice in the literature, we will use it in the rest of this paper, so the correct finite Fourier transform and its inverse are given by Equation (6-6).

The above choice of one of the two special bases described in Subsection 5.4 might at first seem to violate the requirement that both the rules for suppressing the interference and the results should not depend on the particular basis chosen. Actually, the discussion that follows could be formulated in terms of any of the bases allowed by Subsection 5.3, but at the expense of some added complexity in that discussion and in the equations. Since in practice it is desirable to use one of the two bases that are equal to their complex conjugates, it would not serve any purpose to use the more complicated expressions in the rest of this paper. As long as we strictly adhere to the vector-space techniques for interference excision described in Sections 7 and 8, the final result for the vector g^0 with the interference excised is independent of the frequency basis employed. Our use of a special basis is similar to the manner in which an elementary particle is sometimes described in its rest frame or in which two such

particles are described in their center-of-mass frame, even though the theory itself is fully relativistic [17].

The basis frequency Fourier amplitudes are, as in Equation (7-6),

$$\hat{G}(f_k) = \hat{S}(f_k) + \hat{B}(f_k) \; ; \; 0 \leq k \leq N - 1 \; . \tag{9-5}$$

These amplitudes $\hat{G}(f_k)$ satisfy Equation (6-9b) with $\hat{G}(f_0)$ and $\hat{G}(f_{N/2})$ real. Similarly, $\hat{S}(f_0)$, $\hat{S}(f_{N/2})$, $\hat{B}(f_0)$, and $\hat{B}(f_{N/2})$ are real and

$$\hat{S}*(f_k) = \hat{S}(f_{N-k}) \; ; \; 1 \leq k \leq \frac{N}{2} - 1 \; , \tag{9-6a}$$

$$\hat{B}*(f_k) = \hat{B}(f_{N-k}) \; ; \; 1 \leq k \leq \frac{N}{2} - 1 \; . \tag{9-6b}$$

As in Equation (7-7), the corresponding vector in the N-dimensional space C^N is

$$\hat{g} = \hat{s} + \hat{b} \; . \tag{9-7}$$

The discrete Fourier amplitudes $\hat{G}(f_k)$ are related to the continuous amplitudes $G(f)$ by Equation (6-8b), with a similar relation between $\hat{S}(f_k)$ and $S(f)$, as given in Equation (7-9a). However, because of Equation (9-3b), Equation (7-9b) is replaced by

$$\hat{B}(f_k) = \sqrt{N} \left[\int_{F_1}^{F_2} D(f_k - f)B(f)df + \int_{-F_2}^{-F_1} D(f_k - f)B(f)df \right] \; ;$$

$$0 \leq k \leq N - 1 \; . \tag{9-8}$$

Therefore, both the positive frequency integral and the negative frequency one contribute to all of the basis frequency amplitudes.

9.2 MATRIX FORMALISM

Just as we did in Subsection 7.3, we can establish some noise threshold N_o that is much larger than the signal amplitude over the entire frequency range. It is also once again assumed that the interference exceeds this threshold at the basis frequencies in some narrow band given by Equation (7-11), but now, instead of Equation (7-11d), we have

$$\left|\hat{B}(f_k)\right| = \left|\hat{B}(f_{N-k})\right| \geq N_o \; ; \; J \leq k \leq K \; . \tag{9-9a}$$

Equation (7-12) is replaced by the relation

$$\hat{G}(f_k) \simeq \hat{B}(f_k) \; ; \; J \leq k \leq K \; ; \; N - K \leq k \leq N - J \; . \tag{9-9b}$$

We chose L frequencies in the positive frequency band, just as was done in Equations (7-15) and (7-16), and in addition, we must employ the negatives of these frequencies:

$$\phi_p = -\phi_{p-L} \; ; \; L + 1 \leq p \leq 2L \; . \tag{9-10}$$

Equations (7-18) and (7-19) for the positive frequency Fourier amplitudes and Fourier vectors are replaced by

$$\hat{G}(\phi_p) \simeq \hat{B}(\phi_p) \; ; \; 1 \leq p \leq 2L \; , \tag{9-11a}$$

where

$$\hat{G}(\phi_{L+p}) = \hat{G}*(\phi_p) \; ; \; 1 \leq p \leq L \; , \tag{9-11b}$$

and

$$\hat{g}(\phi_p) = \mathcal{V}(\phi_p)\hat{G}(\phi_p) \simeq \mathcal{V}(\phi_p)\hat{B}(\phi_p) \; ; \; 1 \leq p \leq 2L \; , \qquad (9\text{-}12a)$$

where

$$\hat{g}(\phi_{L+p}) = \hat{g}^\star(\phi_p) \; ; \; 1 \leq p \leq L \; . \qquad (9\text{-}12b)$$

Similarly, instead of Equation (7–25), we have

$$\hat{g}(\phi_p) = \sum_{k=0}^{N-1} \mathcal{V}(f_k)\hat{G}(\phi_p,f_k) \; ; \; 1 \leq p \leq 2L \; , \qquad (9\text{-}13a)$$

where

$$\hat{G}(\phi_p,f_k) = D(f_k - \phi_p)G(\phi_p) \; ; \; 0 \leq k \leq N - 1 \; ; \; 1 \leq p \leq 2L \; . \qquad (9\text{-}13b)$$

Our goal, as it was for an interference band in a complex time series, is to find the subspace that best fits the interference, and we can proceed directly to the approach outlined in Section 8. The matrix $\bar{H}$ in Equation (8-9a) is now the $2L \times 2L$ real symmetric matrix

$$\bar{H} = \bar{G}^T\bar{G} \; . \qquad (9\text{-}14)$$

Here the elements of the $N \times 2L$ real matric $\bar{G}$ are given by the following generalization of Equation (8-3b):

$$G_{kp} = \sqrt{\mu_p}D^\circ(f_k - \phi_p)|\hat{G}(\phi_p)| \; ; \; 0 \leq k \leq N - 1 \; ; \; 1 \leq p \leq 2L \; , \qquad (9\text{-}15a)$$

where

$$G_{N-k,L+p} = G_{kp} \; ; \; 1 \leq k \leq \frac{N}{2} \; ; \; 1 \leq p \leq L \; , \tag{9-15b}$$

$$G_{0,L+p} = G_{0,p} \; ; \; 1 \leq p \leq L \; . \tag{9-15c}$$

We have also used the fact that the weighting factors μ_p satisfy

$$\mu_{L+p} = \mu_p \; ; \; 1 \leq p \leq L \; . \tag{9-16}$$

Thus, the elements of $\tilde{H}$ are

$$H_{pq} = \sum_{k=0}^{N-1} G_{kp} G_{kq} \tag{9-17a}$$

$$= \sqrt{\mu_p \mu_q} \, |\hat{G}(\phi_p)| \, D^\circ(\phi_p - \phi_q) \, |\hat{G}(\phi_q)| \; ; \; 1 \leq p,q \leq 2L \; . \tag{9-17b}$$

As was the case in Sections 7 and 8, the eigenvalues of $\tilde{H}$ are real and non-negative, and we order them so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2L} \geq 0 \; . \tag{9-18a}$$

The corresponding eigenvectors $\tilde{V}^\alpha$ are the solutions of the following gener-alization of Equation (8-9b):

$$\tilde{H}\tilde{V}^\alpha = \lambda_\alpha \tilde{V}^\alpha \; ; \; 1 \leq \alpha \leq 2L \; , \tag{9-18b}$$

where the eigenvectors satisfy the orthogonality relations

$$(\tilde{V}^\alpha)^+ \tilde{V}^\beta = \delta_{\alpha\beta} \; ; \; 1 \leq \alpha,\beta \leq 2L \; . \tag{9-18c}$$

9-6

In this case the $\tilde{V}^\alpha$'s are column matrices:

$$\tilde{V}^\alpha = \begin{bmatrix} V_1^\alpha \\ V_2^\alpha \\ . \\ . \\ . \\ V_{2L}^\alpha \end{bmatrix} ; \ 1 \leq \alpha \leq 2L \ .$$

(9-18d)

9.3 CONSTRAINTS ON THE EIGENVECTORS

Because of Equation (9-15), the elements of $\tilde{H}$, defined by Equation (9-17), have the property

$$\left. \begin{array}{lll} H_{L+p,L+q} = H_{pq} & = H_{qp} \\ H_{L+p,q} & = H_{p,L+q} = H_{L+q,p} \end{array} \right\} ; \ 1 \leq p,q \leq L \ .$$

(9-19)

This relation may be written in the form

$$\tilde{H} = \tilde{C}\tilde{H}\tilde{C} \ ,$$

(9-20a)

where the real symmetric matrix $\tilde{C}$ has the elements

$$C_{pq} = \delta_{L+p,q} + \delta_{p,q+L} \ ; \ 1 \leq p,q \leq 2L \ .$$

(9-20b)

and is its own inverse.

When one multiplies both sides of Equation (9-18b) on the left by $\tilde{C}$, the result is

$$H(\tilde{C}\tilde{V}^\alpha) = \lambda_\alpha (\tilde{C}\tilde{V}^\alpha) \ ; \ 1 \leq \alpha \leq 2L \ .$$

(9-21)

Consequently, $C\tilde{V}^\alpha$ is also a solution of Equation (9-18b), and, if the eigenvalue λ_α occurs only once, $C\tilde{V}^\alpha$ must be equal to $\tilde{V}^\alpha$ times a factor ± 1. On the other hand, if λ_α is a multiple root of the eigenvalue equation, we can construct the eigenvectors so that $\tilde{V}^\alpha$ and $C\tilde{V}^\alpha$ are so related. We thus have

$$\tilde{V}^\alpha = \pm C\tilde{V}^\alpha \; ; \; 1 \leq \alpha \leq 2L \; , \tag{9-22a}$$

that is

$$V_{L+p}^\alpha = \pm V_p^\alpha \; ; \; 1 \leq p \leq L \; ; \; 1 \leq \alpha \leq 2L \; . \tag{9-22b}$$

Half of the eigenvectors have the above property with the + sign and the other half have this property with the − sign. We shall return to this matter in Subsection 10.4, where it will be demonstrated that the problem is equivalent to that of finding the eigenvalues and eigenvectors of two L × L real symmetric matrices.

The generalization of Equation (8-10b) for the components of the unit vectors $\hat{u}^\alpha$ that best fit the subspace spanned by the Fourier vectors $\hat{g}(\phi_p)$ in the interference band is

$$\hat{U}^\alpha(f_k) = \frac{1}{\sqrt{\lambda_\alpha}} \zeta(f_k) \sum_{p=1}^{2L} G_{kp} V_p^\alpha \; ; \; 0 \leq k \leq N - 1 \; ; \; 1 \leq \alpha \leq 2L \; , \tag{9-23a}$$

where $\hat{U}^\alpha(f_0)$ and $\hat{U}^\alpha(f_{N/2})$ are real and

$$\tilde{U}^\alpha(f_k) = [\hat{U}^\alpha(f_{N-k})]^* \; ; \; 1 \leq k \leq \frac{N}{2} - 1 \; ; \; 1 \leq \alpha < 2L \; . \tag{9-23b}$$

9-8

Consequently, the vectors

$$\hat{u}^{\alpha} = \sum_{k=0}^{N-1} \mathcal{V}(f_k) \tilde{U}^{\alpha}(f_k) \; ; \; 1 \leq \alpha \leq 2L \; , \tag{9-24a}$$

are real and satisfy

$$\hat{u}^{\alpha} \cdot \hat{u}^{\beta} = \delta_{\alpha\beta} \; ; \; 1 \leq \alpha, \beta \leq 2L \; . \tag{9-24b}$$

Equation (9-23) requires that

$$V_{L+p}^{\alpha} = (V_p^{\alpha})^* \; ; \; 1 \leq p \leq L \; ; \; 1 \leq \alpha \leq 2L \; . \tag{9-25}$$

Consequently, $\tilde{V}^{\alpha}$ is real when it satisfies Equation (9-22) with the + sign and imaginary when it satisfies that equation with the − sign.

9.4 INTERFERENCE EXCISION BY ORTHOGONAL PROJECTION

As in Subsection 7.6, we choose M vectors $\hat{u}^{\alpha}$ belonging to the M largest eigenvalues of $\tilde{H}$ and project the data vector $\hat{g}$ onto the subspace orthogonal to these vectors, obtaining

$$\hat{g}^0 = \hat{g} - \sum_{p=1}^{M} \hat{u}^{\alpha}(\hat{u}^{\alpha} \cdot \hat{g}) \; . \tag{9-26}$$

Equation (7-41) is replaced by

$$\hat{g}^0 = \sum_{k=0}^{N-1} \mathcal{V}(f_k) \hat{G}^0(f_k) \; , \tag{9-27a}$$

9-9

where the components

$$\hat{G}^0(f_k) = \hat{\mathbb{V}}^*(f_k) \cdot \hat{g}^0 \; ; \; 0 \leq k \leq N - 1 \; , \tag{9-27b}$$

are the Fourier amplitudes with the interference suppressed.

Instead of Equation (7-42) we have

$$\hat{G}^0(f_k) = \hat{G}(f_k) - \sum_{\alpha=1}^{M} \hat{U}^\alpha(f_k) \sum_{l=0}^{N-1} [\hat{U}^\alpha(f_l)]^* \hat{G}(f_l) \; ; \; 0 \leq k \leq \frac{N}{2} \; , \tag{9-28a}$$

and the remaining amplitudes can be found from

$$\hat{G}^0(f_k) = [\hat{G}^0(f_{N-k})]^* \; ; \; \frac{N}{2} + 1 \leq k \leq N - 1 \; . \tag{9-28b}$$

Because of Equation (9-23a), we can replace the above equation by

$$\hat{G}^0(f_k) = \hat{G}(f_k) - \sum_{l=0}^{N-1} \Xi_{kl} \hat{G}(f_l) \; ; \; 0 \leq k \leq N - 1 \; , \tag{9-29a}$$

or by

$$\hat{G}^0(f_k) = \hat{G}(f_k) - \sum_{p=1}^{2L} \Upsilon_{kp} \hat{G}(\phi_p) \; ; \; 0 \leq k \leq N - 1 \; . \tag{9-29b}$$

In the above equation

$$\Upsilon_{kq} = \zeta(f_k - \phi_q) \sum_{p=1}^{2L} D^\circ(f_k - \phi_p) T_{pq} \; ; \; 0 \le k \le N - 1 \; ;$$

$$1 \le q \le 2L \; , \tag{9-30a}$$

$$\Xi_{kl} = \sum_{q=1}^{2L} \Upsilon_{kq} D(\phi_q - f_l) \; ; \; 0 \le k, l \le N - 1 \; , \tag{9-30b}$$

where

$$T_{pq} = \sqrt{\mu_p \mu_q} \; |\hat{G}(\phi_p)| \, |\hat{G}(\phi_q)| \sum_{\alpha=1}^{M} \frac{V_p^\alpha (V_q^\alpha)^\star}{\lambda_\alpha} \; ; \; 1 \le p, q \le 2L \; . \tag{9-31}$$

These results generalize Equations (8-11) through (8-14).

As in Subsection 7.6, M is chosen to be the smallest number such that

$$|\hat{G}^0(f_k)| \ll N_o \; ; \; J \le k \le K \; , \tag{9-32}$$

for the basis frequencies in the positive frequency part of the band. The corresponding relations for the negative frequencies will then be satisfied. (See also the comments following Equation (7-44).) Then the time series

$$g_k^0 \; , \; 0 \le k \le N - 1 \; , \tag{9-33a}$$

with the interference suppressed is found from the inverse finite Fourier transform:

9-11

$$g_k^0 = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \hat{G}^0(f_l) e^{j2\pi lk/N} \; ; \; 0 \leq k \leq N - 1 \; . \qquad (9\text{--}33b)$$

For the reasons discussed at the end of Subsection 8.2, M is expected to be approximately equal to $2(K - J + 1)$, the number of basis frequencies in the interference band.

SECTION 10

EXCISION OF MULTIPLE INTERFERENCE BANDS

10.1 GENERALIZATION OF PREVIOUS RESULTS

The results of the last section can now easily be generalized to include any number of interference bands in a real time series. Some relations will be given for the real symmetric $2L \times 2L$ matrix $\tilde{H}$ whose largest eigenvalues and corresponding eigenvectors must be found and also for its submatrices. Then it will be shown that the problem is easily transformed into one involving two $L \times L$ matrices.

We can continue to write the real finite time series in the form of Equation (9-4). The finite Fourier transform satisfies Equations (9-5) and (9-6), and the corresponding vector in the N-dimensional space C^N is given by Equation (9-7). However, Equation (9-8) is replaced by one with the required number of integrals over the different interference bands.

As was the case earlier, we establish a noise threshold N_o that is much larger than the signal amplitude over the frequency range. Suppose that we are able to distinguish I different bands such that in the positive frequency region, Equation (9-9) generalizes to

$$\left. \begin{array}{l} \hat{G}(f_k) \simeq \hat{B}(f_k) \\[2mm] |\hat{G}(f_k)| \geq N_o \end{array} \right\} \; ; \; J_a \leq k \leq K_a \; ; \; 1 \leq a \leq I \; , \tag{10-1a}$$

where

$$\frac{N}{2} > K_I \geq J_I > K_{I-1} \geq J_{I-1} > \; . \; . \; . \; > K_1 \geq J_1 > 0 \; . \tag{10-1b}$$

For the negative frequencies the amplitudes are the complex conjugates of the corresponding positive frequency ones.

Just as we did in Equations (7-16) and (7-17), we choose L_a frequencies ϕ_p in positive frequency band a, where

$$N \gg L_a \gtrsim 4(K_a - J_a) + 1 \; ; \; 1 \leq a \leq I \; . \qquad (10\text{-}2a)$$

The total number of such positive frequencies in the interference bands is

$$L = \sum_{a=1}^{I} L_a \; . \qquad (10\text{-}2b)$$

Let us also define

$$\bar{L}_o = 0 \; , \qquad (10\text{-}3a)$$

$$\bar{L}_a = \sum_{b=1}^{a} L_b \; ; \; 1 \leq a \leq I - 1 \; , \qquad (10\text{-}3b)$$

$$\bar{L}_I = L \; . \qquad (10\text{-}3c)$$

Thus,

$$\text{positive frequency band a} = \{\phi_p : \bar{L}_{a-1} + 1 \leq p \leq \bar{L}_a\} \; . \qquad (10\text{-}4)$$

As in Equation (9-10), the negative interference frequencies are:

$$\phi_p = -\phi_{p-L} \; ; \; L + 1 \leq p \leq 2L \; , \qquad (10\text{-}5a)$$

10-2

so

negative frequency band a =

$$= \{ \phi_p : L + \bar{L}_{a-1} + 1 \leq p \leq L + \bar{L}_a \} . \tag{10-5b}$$

The procedure outlined in Equations (9-11) through (9-33) for suppression of a single narrow band in a real time series is still valid, the only exception being that Equation (9-32) is replaced by the requirement that M be the smallest number such that

$$\left| \hat{G}^0(f_k) \right| \ll N_o ; \quad J_a \leq k \leq K_a ; \quad 1 \leq a \leq I . \tag{10-6}$$

We now expect M to be approximately equal to

$$2 \sum_{a=1}^{I} (K_a - J_a + 1) .$$

Instead of trying to excise all of the interference at once, it is probably best to initially assign a very high value to N_o, because the strongest interferers may have sidelobes that exceed or are comparable in magnitude with the amplitudes of less powerful interferers. Once the interference bands with the largest amplitudes have been excised, one can lower the value of N_o and repeat the procedure a number of times. In this way the real symmetric 2L × 2L matrix whose largest eigenvalues and corresponding eigenvectors must be found can be kept small enough for practical calculations; i.e., one can assume that L << N.

10.2 DECOMPOSITION OF THE MATRIX $\tilde{H}$

Equation (9-17b) is a particularly useful form in which to write the elements of the $2L \times 2L$ real symmetric matrix $\tilde{H}$. Let us define the real symmetric matrix

$$\tilde{D} = \tilde{D}^T \tag{10-7a}$$

by

$$D_{pq} = D°(\phi_p - \phi_q) \; ; \; 1 \leq p,q \leq 2L \; , \tag{10-7b}$$

and the diagonal matrix $\tilde{\Gamma}$ by

$$\Gamma_{pq} = \delta_{pq} \; \sqrt{\mu_p} \, |\hat{G}(\phi_p)| ; \; 1 \leq p,q \leq 2L \; . \tag{10-8}$$

Then

$$\tilde{H} = \tilde{\Gamma} \, \tilde{D} \, \tilde{\Gamma} \; . \tag{10-9}$$

Note that, as was the case for $\tilde{H}$ in Equation (9-20),

$$\tilde{D} = \tilde{C} \, \tilde{D} \, \tilde{C} \; , \tag{10-10a}$$

$$\tilde{\Gamma} = \tilde{C} \, \tilde{\Gamma} \, \tilde{C} \; . \tag{10-10b}$$

The matrix $\tilde{C}$ may be expressed in the form

$$\tilde{C} = \begin{bmatrix} \tilde{0}_L & \tilde{I}_L \\ \tilde{I}_L & \tilde{0}_L \end{bmatrix} \; , \tag{10-11}$$

10-4

where $\tilde{I}_L$ and $\tilde{0}_L$ are, respectively, the L × L unit and zero matrices. Similarly, the matrices $\tilde{H}$, $\tilde{D}$, and $\tilde{\Gamma}$ can be written

$$\tilde{H} = \begin{bmatrix} \tilde{H}(+) & \tilde{H}(-) \\ \tilde{H}(-) & \tilde{H}(+) \end{bmatrix} , \tag{10-12a}$$

$$\tilde{D} = \begin{bmatrix} \tilde{D}(+) & \tilde{D}(-) \\ \tilde{D}(-) & \tilde{D}(+) \end{bmatrix} , \tag{10-12b}$$

$$\tilde{\Gamma} = \begin{bmatrix} \tilde{\Gamma}(+) & \tilde{0}_L \\ \tilde{0}_L & \tilde{\Gamma}(+) \end{bmatrix} , \tag{10-12c}$$

where all of the L × L submatrices are real and symmetric.

According to Equation (10-9),

$$\tilde{H}(\pm) = \tilde{\Gamma}(+)\tilde{D}(\pm)\tilde{\Gamma}(+) . \tag{10-13}$$

From Equations (9-17) and (10-5a),

$$H_{pq}(\pm) = \sqrt{\mu_p \mu_q} \ |\hat{G}(\phi_p)| \ D^\circ(\phi_p \mp \phi_q) \ |\hat{G}(\phi_q)| \tag{10-14a}$$

$$= H_{qp}(\pm) \ ; \ 1 \leq p,q \leq L . \tag{10-14b}$$

Similarly, according to Equations (10-7) and (10-8),

$$D_{pq}(\pm) = D_{qp}(\pm) = D^\circ(\phi_p \mp \phi_q) \ ; \ 1 \leq p,q \leq L , \tag{10-15a}$$

and

$$\Gamma_{pq}(+) = \delta_{pq}\sqrt{\mu_p}\,|\hat{G}(\phi_p)| \;\; ; \;\; 1 \le p,q \le L \; . \tag{10-15b}$$

10.3 FURTHER DECOMPOSITION OF $\tilde{H}$

Let us employ the notation

$$\phi_p^a = \phi_{\overline{L}_{a-1}+p} \;\; ; \;\; 1 \le p \le L_a \; , \tag{10-16}$$

to designate the frequencies in positive frequency band a, where $1 \le a \le I$.

The real submatrix $\tilde{H}(+)$ above may be written:

$$\tilde{H}(+) = \begin{bmatrix} \tilde{H}(1,1) & \tilde{H}(1,2) & . & . & . & \tilde{H}(1,I) \\ \tilde{H}(2,1) & \tilde{H}(2,2) & . & . & . & \tilde{H}(2,I) \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ \tilde{H}(I,1) & \tilde{H}(I,2) & . & . & . & \tilde{H}(I,I) \end{bmatrix} , \tag{10-17}$$

where, from Equation (10-14), the elements of the $L_a \times L_b$ sub-submatrix $\tilde{H}(a,b)$ are

$$H_{pq}(a,b) = \sqrt{\mu_p \mu_q}\,|\hat{G}(\phi_p^a)|\; D^{\circ}(\phi_p^a - \phi_q^b)\,|\hat{G}(\phi_q^b)| \tag{10-18a}$$

$$= H_{qp}(b,a) \;\; ; \;\; 1 \le p \le L_a \;\; ; \;\; 1 \le q \le L_b \; . \tag{10-18b}$$

Therefore,

$$\tilde{H}(b,a) = \tilde{H}^T(a,b) \; ; \; 1 \leq a,b \leq I \; , \tag{10-19}$$

so the matrices $\tilde{H}(a,a)$ on the diagonal of $\bar{H}(+)$ are symmetric.

Similarly, the submatrix $\bar{H}(-)$ has the form

$$\tilde{H}(-) = \begin{bmatrix} \tilde{J}(1,1) & \tilde{J}(1,2) & \cdot & \cdot & \cdot & \tilde{J}(1,I) \\ \tilde{J}(2,1) & \tilde{J}(2,2) & \cdot & \cdot & \cdot & \tilde{J}(2,I) \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \tilde{J}(I,1) & \tilde{J}(I,2) & \cdot & \cdot & \cdot & \tilde{J}(I,I) \end{bmatrix} \; , \tag{10-20}$$

where the elements of the $L_a \times L_b$ sub-submatrix $\tilde{J}(a,b)$ are given by

$$J_{pq}(a,b) = \sqrt{\mu_p \mu_q} \; |\hat{G}(\phi_p^a)| \; D^\circ(\phi_p^a + \phi_p^a) \; |\hat{G}(\phi_q^b)| \tag{10-21a}$$

$$= J_{qp}(b,a) \; ; \; 1 \leq p \leq L_a \; ; \; 1 \leq q \leq L_b \; . \tag{10-21b}$$

As a result,

$$\tilde{J}(b,a) = \tilde{J}^T(a,b) \; ; \; I \leq a,b \leq I \; , \tag{10-22}$$

so the matrices $\tilde{J}(a,a)$ on the diagonal of $\bar{H}(-)$ are symmetric.

Just as we did for $\bar{H}(+)$, we can write $\bar{D}(+)$ in the form

$$\tilde{D}(+) = \begin{bmatrix} \tilde{D}(1,1) & \tilde{D}(1,2) & . & . & . & \tilde{D}(1,I) \\ \tilde{D}(2,1) & \tilde{D}(2,2) & . & . & . & \tilde{D}(2,I) \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ \tilde{D}(I,1) & \tilde{D}(I,2) & . & . & . & \tilde{D}(I,I) \end{bmatrix} , \qquad (10\text{--}23)$$

where the elements of the $L_a \times L_b$ sub-submatrix $\tilde{D}(a,b)$ are

$$D_{pq}(a,b) = D^\circ(\phi_p^a - \phi_q^b) \qquad (10\text{--}24a)$$

$$= D_{qp}(b,a) \; ; \; 1 \le p \le L_a \; ; \; 1 \le q \le L_b \; . \qquad (10\text{--}24b)$$

Here we have

$$\tilde{D}(b,a) = \tilde{D}^T(a,b) \; ; \; 1 \le a,b \le I \; , \qquad (10\text{--}25)$$

so the matrices $\tilde{D}(a,a)$ on the diagonal of $\tilde{D}(+)$ are symmetric.

Similarly, one can write $\tilde{D}(-)$ in a form like that in Equation (10-20):

$$\tilde{D}(-) = \begin{bmatrix} \tilde{E}(1,1) & \tilde{E}(1,2) & . & . & . & \tilde{E}(1,I) \\ \tilde{E}(2,1) & \tilde{E}(2,2) & . & . & . & \tilde{E}(2,I) \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ \tilde{E}(I,1) & \tilde{E}(I,2) & . & . & . & \tilde{E}(I,I) \end{bmatrix} , \qquad (10\text{--}26)$$

where

$$E_{pq}(a,b) = D(\phi_p^a + \phi_q^b) \tag{10-27a}$$

$$= E_{qp}(b,a) \; ; \; 1 \leq p \leq L_a \; ; \; 1 \leq q \leq L_b \; . \tag{10-27b}$$

Once again we have the property

$$\tilde{E}(b,a) = \tilde{E}^T(a,b) \; ; \; 1 \leq a,b \leq I \; , \tag{10-28}$$

and the matrices $\tilde{E}(a,a)$ along the diagonal of $\tilde{D}(-)$ are symmetric.

Finally, let us write for $\tilde{\Gamma}(+)$:

$$\tilde{\Gamma}(+) = \begin{bmatrix} \tilde{\Gamma}(1) & \tilde{0}(1,2) & \cdots & \tilde{0}(1,I) \\ \tilde{0}(2,1) & \tilde{\Gamma}(2) & \cdots & \tilde{0}(2,I) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \tilde{0}(I,1) & \tilde{0}(I,2) & \cdots & \tilde{\Gamma}(I) \end{bmatrix} , \tag{10-29a}$$

where $\tilde{0}(a,b)$ is the $L_a \times L_b$ zero matrix and where the diagonal matrix $\tilde{\Gamma}(a)$ has the elements

$$\Gamma_{pq}(a) = \delta_{pq} \sqrt{\mu_p} \, |\hat{G}(\phi_p)| ; \; 1 \leq p,q \leq L_a \; . \tag{10-29b}$$

Then, according to Equation (10-13),

$$\tilde{H}(a,b) = \tilde{\Gamma}(a)\tilde{D}(a,b)\tilde{\Gamma}(b) \; ; \; 1 \leq a,b \leq I \; , \tag{10-30a}$$

$$\tilde{J}(a,b) = \tilde{\Gamma}(a)\tilde{E}(a,b)\tilde{\Gamma}(b) \; ; \; 1 \leq a,b \leq I \; . \tag{10-30b}$$

10.4 TRANSFORMATION TO L × L MATRICES

Let us introduce the unitary matrix

$$\tilde{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{I}_L & \tilde{I}_L \\ -j\tilde{I}_L & j\tilde{I}_L \end{bmatrix} , \tag{10-31a}$$

the inverse of which is

$$\tilde{S}^{-1} = \tilde{S}^{+} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{I}_L & j\tilde{I}_L \\ \tilde{I}_L & -j\tilde{I}_L \end{bmatrix} . \tag{10-31b}$$

A similarity transformation on $\tilde{H}$ with this matrix yields the equivalent real symmetric matrix

$$\tilde{R} = \tilde{S}\tilde{H}\tilde{S}^{-1} , \tag{10-32a}$$

which has the form

$$\tilde{R} = \begin{bmatrix} \tilde{R}(+) & \tilde{0} \\ \tilde{0} & \tilde{R}(-) \end{bmatrix} , \tag{10-32b}$$

where

$$\tilde{R}(\pm) = \tilde{H}(+) \pm \tilde{H}(-) . \tag{10-32c}$$

Thus, the problem of finding the eigenvalues and eigenvectors of the 2L × 2L matrix $\tilde{H}$ has been simplified to the corresponding problem for the two L × L matrices $\tilde{R}(+)$ and $\tilde{R}(-)$.

In fact, when Equation (9-18b) is multiplied on the left by $\tilde{S}$ and use is made of Equation (10-32a), we find that

$$\tilde{R}\tilde{W}^\alpha = \lambda_\alpha \tilde{W}^\alpha \; ; \; 1 \leq \alpha \leq 2L \; , \tag{10-33}$$

where

$$\tilde{W}^\alpha = \tilde{S}\tilde{V}^\alpha \; ; \; 1 \leq \alpha \leq 2L \; , \tag{10-34a}$$

has the form

$$\tilde{W}^\alpha = \begin{bmatrix} \tilde{W}^\alpha(+) \\ \tilde{W}^\alpha(-) \end{bmatrix} \; ; \; 1 \leq \alpha \leq 2L \; . \tag{10-34b}$$

In the above $\tilde{W}^\alpha(\pm)$ is an $L \times 1$ column matrix. Because of Equation (10-32b),

$$\tilde{R}(\pm)\tilde{W}^\alpha(\pm) = \lambda_\alpha \tilde{W}^\alpha(\pm) \; ; \; 1 \leq \alpha \leq 2L \; . \tag{10-35}$$

It follows that each of the eigenvalues λ_α of $\tilde{H}$ is either an eigenvalue of $\tilde{R}(+)$ or an eigenvalue of $\tilde{R}(-)$. In the former case,

$$\tilde{R}(+)\tilde{W}^\alpha(+) = \lambda_\alpha \tilde{W}^\alpha(+) \; , \tag{10-36a}$$

$$\tilde{W}^\alpha(-) = 0 \; , \tag{10-36b}$$

whereas in the latter case

$$\tilde{R}(-)\tilde{W}^\alpha(-) = \lambda_\alpha \tilde{W}^\alpha(-) \; , \tag{10-37a}$$

$$\tilde{W}^\alpha(+) = 0 \; . \tag{10-37b}$$

10-11

The inverse of Equation (10-34a) is

$$\tilde{V}^{\alpha} = \tilde{S}^{-1}\tilde{W}^{\alpha} \;\; ; \; 1 \leq \alpha \leq 2L \;\;. \tag{10-38}$$

If λ_{α} is an eigenvalue of $\tilde{R}(+)$, we then have

$$\tilde{V}^{\alpha} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{W}^{\alpha}(+) \\ \tilde{W}^{\alpha}(-) \end{bmatrix} \;\;, \tag{10-39a}$$

But, if λ_{α} is an eigenvalue of $\tilde{R}(-)$,

$$\tilde{V}^{\alpha} = \frac{1}{\sqrt{2}} \begin{bmatrix} j\tilde{W}^{\alpha}(-) \\ -j\tilde{W}^{\alpha}(-) \end{bmatrix} \;\;. \tag{10-39b}$$

If we choose the eigenvectors $\tilde{W}^{\alpha}(\pm)$ to be real, which is always possible, then the eigenvectors in Equation (10-39a) are real and satisfy Equation (9-22) with the + sign on the right, while the eigenvectors in Equation (10-39b) are imaginary and satisfy Equation (9-22) with the − sign on the right -- exactly half of the 2L eigenvectors of $\tilde{H}$ have the former properties and the other half have the latter properties. This result is in agreement with the properties of the eigenvectors of $\tilde{H}$ that were discussed in Subsection 9.3.

Because of Equation (10-13) we can write

$$\tilde{R}(\pm) = \tilde{\Gamma}(+)\left[\tilde{D}(+) \pm \tilde{D}(-)\right]\tilde{\Gamma}(+) \;\;. \tag{10-40a}$$

10-12

That is,

$$R_{pq}(\pm) = \sqrt{\mu_p \mu_q} \, |\hat{G}(\phi_p)| \left[D^\circ(\phi_p - \phi_q) \pm D^\circ(\phi_p + \phi_q) \right] |\hat{G}(\phi_q)| \; ;$$

$$1 \leq p,q \leq L \; . \qquad (10-40b)$$

The results just obtained are useful when employed in association with Equations (9-29) through (9-31) for the Fourier amplitudes $G^0(f_k)$ with the narrowband interference suppressed. It was pointed out in connection with those equations that we need only the M largest eigenvalues of $\tilde{H}$ and their corresponding eigenvectors such that Equation (9-32) is satisfied. Let us suppose that, of these M largest eigenvalues, M_+ are eigenvalues $\lambda_\alpha(+)$ of $\tilde{R}(+)$ and M_- are eigenvalues $\lambda_\alpha(-)$ of $\tilde{R}(-)$, where

$$M = M_+ + M_- \; . \qquad (10-41)$$

Then, Equation (9-29b) can be written

$$\hat{G}^0(f_k) = \hat{G}(f_k) - \sum_{p=1}^{L} \left[T_{kp}(+)\hat{G}(\phi_p) + T_{kp}(-)\hat{G}*(\phi_p) \right] \; ; \; 0 \leq k \leq \frac{N}{2} \; , \qquad (10-42a)$$

$$\hat{G}^0(f_k) = \hat{G}*(f_{N-k}) \; ; \; \frac{N}{2} + 1 \leq k \leq N - 1 \; . \qquad (10-42b)$$

Here

$$T_{kp}(\pm) = \zeta(f_k \mp \phi_p) \sum_{q=1}^{L} \left[D^\circ(f_k - \phi_q)T_{qp}(\pm) + D^\circ(f_k + \phi_q)T_{qp}(\mp) \right] \; ;$$

$$0 \leq k \leq N - 1 \; ; \; 1 \leq p \leq L \; , \qquad (10-43)$$

where

$$T_{qp}(\pm) = \sqrt{\mu_p \mu_q} \, |\hat{G}(f_p)| \, |\hat{G}(\phi_q)| \left[\mathfrak{Q}_{qp}(+) \pm \mathfrak{Q}_{qp}(-) \right] \; ; \; 1 \leq p,q \leq L \; , \tag{10-44a}$$

$$\mathfrak{Q}_{qp}(\pm) = \sum_{\alpha=1}^{M_{\pm}} \frac{W_p^{\alpha}(\pm) W_q^{\alpha}(\pm)}{\lambda_{\alpha}(\pm)} \; ; \; 1 \leq p,q \leq L \; . \tag{10-44b}$$

SECTION 11

SUMMARY AND DISCUSSION

The main results of this paper are as follows:

a. The N measured values of a band-limited time series (e.g., a radio
    signal plus interference) that has been sampled at the Nyquist
    rate can be regarded as the components of a vector in an N-
    dimensional space. Each frequency in the continuous range of
    frequencies less than the Nyquist frequency is represented by a
    specific unit vector in this space, but only N of these vectors
    are linearly independent. Any subset of N orthonormal frequency
    vectors can be used as a basis (set of coordinate axes). Multi-
    plying the unit vector for a particular frequency by the finite
    Fourier transform for that frequency yields a Fourier vector.

b. Filters to suppress narrowband interference should not depend on
    the particular frequency basis chosen; that is, the rules for
    designing them and the results obtained should be the same (at
    least, within a reasonable approximation) regardless of the fre-
    quency coordinates employed. The goal should be to excise the
    subspace that best fits the continuous range of Fourier vectors in
    a narrow interference band, thus accounting for the sidelobes of
    the band.

c. To find the desired subspace for an arbitrary number of narrowband
    interferers, this paper has employed a least squares approach.
    One looks for the unit vector such that the sum of the squared
    absolute values of the projections of the Fourier vectors in these
    bands onto the subspace orthogonal to that unit vector is a mini-
    mum. This requires finding the largest eigenvalues and the

corresponding eigenvectors of an N × N Hermitian matrix. (To suppress the interference, the data vector is projected onto the subspace orthogonal to these eigenvectors.) A simple transformation makes it possible to reformulate the eigenvalue and eigenvector problem in terms of much smaller real symmetric L × L matrices, where L ≪ N.

This paper has given a theoretical treatment of the narrowband interference excision problem and has not discussed the practical application of the vector space methods described here. Although considerable improvement in performance is expected over conventional frequency-domain methods for suppressing narrowband interference, the degree of this improvement is presently unknown, and testing must be performed. (The signal-to-interference improvements using vector space methods should be particularly significant in a high noise environment with many closely-spaced strong narrowband interferers.) The specific system and hardware requirements to implement vector space signal processing methods must be identified and compared to the requirements for known frequency-domain and time-domain approaches. Although these requirements will certainly be greater for vector space methods, it is believed that the processors becoming available will make the digital processing feasible in real time.

Many other questions remain to be addressed. For example, how many frequencies in a narrowband (the frequencies ϕ_p in Equations (7-16) and (10-4), that is) should be used, when proper consideration is given to the degree of accuracy desired and to the computational requirements? This question is part of the problem of determining the numerical method to be employed in approximating the integral in Equation (7-23), including the choice of the weighting factors μ_p.

The application of the vector space approach to narrowband interference suppression is not necessarily restricted to the least squares

technique discussed in this paper, and approximations requiring less signal processing should also be investigated. It should be possible to reformulate more conventional frequency-domain filter design techniques in terms of the results in Sections 5 and 6 and the principle that such techniques should be independent of the particular frequency basis selected. This would provide a better theoretical understanding of such filters and suggest methods for improving them.

The motivation for the investigation reported in this paper was the excision of strong narrowband interference in a pseudonoise spread-spectrum communication system, in which case the spectrum of the PN sequence itself is relatively flat and such interference is easy to recognize. It should be possible to adapt the techniques described here to other systems in which strong narrowband interference is present.

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APPENDIX A

EVALUATION OF AN INTEGRAL

It was pointed out in Subsection 7.4 that Equation (7-22b), which one seeks to maximize by the proper choice of the unit vector $\hat{u}$, should be regarded as an approximation for the integral in Equation (7-23). That integral will be evaluated in this appendix, and it will then be shown, by comparing the result to that given by Equation (7-22b), how we obtain the restriction

$$L \geq 4(K - J) + 1 \tag{A-1}$$

in Equation (7-17).

The same steps that led to Equation (7-27) make it possible to continue to write Λ in that form, where now, however, in place of Equation (7-28) one has

$$K_{kl} = \int_{\overline{F}_1}^{\overline{F}_2} df \; D(\overline{f}_k - f) |\hat{X}(f)|^2 D(f - \overline{f}_1) \; ; \; 0 \leq k, l \leq N - 1 \; . \tag{A-2}$$

Because of Equations (5-4) and (6-11b),

$$K_{kl} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \exp\left[-j(N - 1)\pi T_0(\overline{f}_i - \overline{f}_j + \overline{f}_k - \overline{f}_1)\right] \hat{X}^\star(\overline{f}_i) Q^\circ_{ijkl} \hat{X}(\overline{f}_j) \; ;$$

$$0 \leq k, l \leq N - 1 \; , \tag{A-3}$$

where

$$Q^\circ_{ijkl} = \int\limits_{\overline{F}_1}^{\overline{F}_2} df \ D^\circ(f - \overline{f}_i)D^\circ(f - \overline{f}_j)D^\circ(f - \overline{f}_k)D^\circ(f - \overline{f}_l) \ ;$$

$$0 \leq i,j,k,l \leq N - 1 \ . \qquad (A-4)$$

The identity

$$\frac{1}{\sin a \ \sin b} = - \ \frac{\cot a - \cot b}{\sin (a - b)}$$

leads to the result

$$[\sin(x - x_i) \ \sin(x - x_j) \ \sin (x - x_k) \ \sin(x - x_l)]^{-1} =$$

$$= A(i;j,k,l) \ \cot(x - x_i) + A(j;i,k,l) \ \cot(x - x_j)$$

$$+ A(k;i,j,l) \ \cot(x - x_k) + A(l;i,j,k) \ \cot(x - x_l) \ , \quad (A-5a)$$

where

$$A(i;j,k,l) = [\sin(x_i - x_j) \ \sin(x_i - x_k) \ \sin(x_i - x_l)]^{-1} \ . \qquad (A-5b)$$

When Equations (5-3), (5-4), and (A-5) are substituted into Equation (A-4), the result is

$$Q^0_{ijkl} = \frac{1}{8N^4} \int\limits_{\overline{F}_1}^{\overline{F}_2} df \ [P(f;i;j,k,l) + P(f;j;i,k,l)$$

$$+ P(f;k;i,j,l) + P(f;l;i,j,k)] \ . \qquad (A-6)$$

A-2

In the above expression

$$P(f;i;j,k,l) = A(i;j,k,l) \sum_{m=1}^{4} \sum_{n=1}^{2N} \alpha_m(n) \times$$

$$\times \sin\{2\pi n T_0(f - \overline{f}_i) + N\beta_m(i;j,k,l)\} , \qquad (A-7)$$

where

$$\alpha_1(n) = \alpha_2(n) = \alpha_3(n) = \begin{cases} 2 \ ; \ 1 \le n \le N - 1 \ , \\ 1 \ ; \qquad n = N \qquad , \\ 0 \ ; \ N + 1 \le n \le 2N \ , \end{cases} \qquad (A-8a)$$

$$\alpha_4(n) = \begin{cases} 0 \ ; \ 1 \le n \le N - 1 \ , \\ -1 \ ; \qquad n = N \qquad , \\ -2 \ ; \ N + 1 \le n \le 2N \ , \end{cases} \qquad (A-8b)$$

and

$$\beta_1(i;j,k,l) = \pi T_0(\overline{f}_i + \overline{f}_j - \overline{f}_k - \overline{f}_l) , \qquad (A-9a)$$

$$\beta_2(i;j,k,l) = \pi T_0(\overline{f}_i - \overline{f}_j + \overline{f}_k - \overline{f}_l) , \qquad (A-9b)$$

$$\beta_3(i;j,k,l) = \pi T_0(\overline{f}_i - \overline{f}_j - \overline{f}_k + \overline{f}_l) , \qquad (A-9c)$$

$$\beta_4(i;j,k,l) = \pi T_0(3\overline{f}_i - \overline{f}_j - \overline{f}_k - \overline{f}_l) . \qquad (A-9d)$$

After carrying out the integration in Equation (A-6), we obtain

$$Q^0_{ijkl} = \frac{1}{8N^4}[R(i;j,k,l) + R(j;i,k,l) + R(k;i,j,l) + R(l;i,j,k)] , \qquad (A-10)$$

A-3

where

$$R(i;j,k,l) = A(i;j,k,l) \sum_{m=1}^{4} \sum_{n=1}^{2N} \alpha_m(n) \frac{\sin[\pi T_0 n(\overline{F}_2 - \overline{F}_1)]}{\pi T_0 n} \times$$

$$\times \sin\{\pi T_o[n(\overline{F}_2 + \overline{F}_1) - 2n\overline{f}_i] + N\beta_m(i;j,k,l)\} \quad . \quad (A\text{-}11)$$

Equations (A-3), (A-5b), and (A-8) through (A-11) give us the elements of the matrix $\tilde{K}$ when the integration is exact.

Suppose we decide to evaluate the approximation in Equation (7-22b) by using equally spaced interference frequencies ϕ_p in Equation (7-16). To simplify, we shall assume that $\overline{F}_2$ and $\overline{F}_1$ in Equation (7-14) coincide with basis frequencies; i.e., in Equation (7-11),

$$\overline{F}_2 = \overline{f}_K \quad , \tag{A-12a}$$

$$\overline{F}_1 = \overline{f}_J \quad . \tag{A-12b}$$

The following results are easily extended to the general case in which this assumption is not valid.

Let us divide the intervals between basis frequencies in the inter-ference band into P equal parts, so that in Equation (7-16)

$$L = P(K - J) + 1 \quad . \tag{A-13}$$

Thus, from Equations (5-17) and (5-18),

$$\phi_p = \overline{f}_J + \frac{(p - 1)F_0}{NP}$$

$$= \bar{f} + \frac{F_0}{N}\left(J + \frac{p-1}{P}\right) \; ; \; 1 \leq p \leq P(K - J) + 1 \; . \tag{A-14}$$

We can still write the elements of $\tilde{K}$ as in Equation (A-3), but now, instead of Equation (A-6), we have

$$Q^0_{ijkl} = \frac{1}{8N^4} \sum_{p=0}^{P(K-J)+1} \mu_p \left[P(\phi_p;i;j,k,l) + P(\phi_p;j;i,k,l) \right.$$

$$\left. + P(\phi_p;k;i,j,l) + P(\phi_p;l;i,j,k) \right] \; , \tag{A-15}$$

The simplest possible numerical integration scheme involves setting the weighting factors μ_p equal to the values given in Equation (7-24c). The result has the form of Equation (A-10), but instead of Equation (A-11) we now have

$$R(i;j,k,l) = A(i;j,k,l) \sum_{m=1}^{4} \sum_{n=1}^{2N} \alpha_m(n) \frac{\sin[\pi T_0 n(\bar{F}_2 - \bar{F}_1)]}{T_0 NP \sin(\pi n/NP)} \times$$

$$\times \sin \left\{ \pi T_0 [n(\bar{F}_2 + \bar{F}_1) - 2n\bar{f}_i] + N\beta_m(i;j,k,l) \right\} \; . \tag{A-16}$$

In practice we expect to have approximately 10^4 pseudonoise sequence bits equal to one data bit, so we should have

$$N \geq 10^4 \; .$$

Therefore, in Equation (A-11), which is the exact result, almost all of the contribution comes from the smallest values of n and for $n > 10^3$ the terms are small. For such values of n

$$\frac{\pi n}{NP} \leq \frac{\pi}{10P} \quad .$$

If

$$P \geq 4 \, , \hspace{6cm} \text{(A-17)}$$

we can use the approximation

$$\sin\left(\frac{\pi n}{NP}\right) \sim \frac{\pi n}{NP} \, ; \, n \leq 10^3$$

in Equation (A-16). Under these conditions the nth terms in Equation (A-11) and (A-16) are approximately equal. Furthermore, the above choice for P guarantees that

$$\frac{\pi n}{NP} \leq \frac{\pi}{2}$$

for all values of n in Equation (A-16), with the equality holding only for n = 2N. Consequently,

$$\sin\left(\frac{\pi n}{NP}\right)$$

always increases in value as n increases. (If P is smaller than 4, the larger values of n might give significant contributions to Equation (A-16) and the result there would not be a good approximation to Equation (A-11).) Thus, Equations (A-13) and (A-17) give us the result in Equation (7-17).